Assignment no. 1

The exercises vary in difficulty, some are easy and some are rather hard. It's improbable that you will solve them all in 1.5 hours. Don't hesitate to ask for more hints than supplied below.

Notation: $\mathcal{L}$ is a set of n lines in general position in the plane, and $\mathcal{A}(\mathcal{L})$ is their arrangement.

Exercise 1.1 A subset T of the plane is *convex* if for any pair of points $p, q \in T$ the line segments $\overline{pq}$ is contained in T.
(a) Show that any face in a planar arrangement of lines is convex.
(b) What is the maximum complexity of a face in an arrangement of n lines.

Exercise 1.2 Let S be a set of n segments in the plane. A line ℓ that intersects all segments of S is called a *transversal* for S. Give an $O(n^2)$ algorithm to decide if a transversal exists for S. **Hint:** Characterize the dual of a line segment.

Exercise 1.3 Give an $O(n \log n)$ algorithm to find the smallest axis-parallel rectangle bounding all the vertices of $\mathcal{A}(\mathcal{L})$.

Exercise 1.4 Let F be the set of faces of $\mathcal{A}(\mathcal{L})$. For a face $f \in F$, let e_f denote the number of edges on its boundary. Show that

$$\sum_{f \in F} e_f^2 = O(n^2).$$

Exercise 1.5 There are n Jeeps driving in a flat desert, each in a straight lane and fixed velocity (the lanes or velocities are not necessarily the same). A spectator stands in a fixed position in the desert. A *proximity interval* is a maximal continuous time interval in which a certain Jeep is **closest** to the spectator. Show that the maximum number of proximity intervals is $2n - 1$, both upper and lower bounds. **Hint:** For each Jeep, consider the graph of the function of its distance to the spectator. Argue about the lower envelope of these graphs.

Exercise 1.6, Open Problem (beware!) If you are familiar with computational geometry and arrangements, and have solved the other exercises, you may want to look into the following problem.
Let P be a simple polygon with n vertices. We denote by V the set of vertices of P. For a directed line λ in the plane, we denote by $h_l(\lambda)$ (resp. $h_r(\lambda)$) the open half-plane bounded by λ on the left- (resp. right-) hand-side of λ. The line λ is an *area bisector* of P if the area of $P \cap h_l(\lambda)$ is equal to the area of $P \cap h_r(\lambda)$.
A line λ partitions V into three sets (two of which may be empty): $V \cap h_l(\lambda)$, $V \cap \lambda$, and $V \cap h_r(\lambda)$. We say that two area bisectors of P are *combinatorially distinct* if the partitioning of V as above induced by the two bisectors is different. We say that two area bisectors of P are *combinatorially equivalent* if they induce the same partitioning of V.
For every orientation $\theta \in [0, \pi)$ there is a single area bisector with orientation θ. A *bisector interval* is a maximal continuous interval of $[0, \pi)$ in which all the bisectors are combinatorially equivalent.
What is the maximum number of bisector intervals in $[0, \pi)$ over all simple polygons with n vertices? The currently known bounds are: lower bound $\Omega(n^2)$, and upper bound $O(n^3)$.