Advanced Course on the Foundations of Computer Science 2014

Note on Multiplicative-Weights Update

Lecturer: Aleksander Mądry

Recall that in our setup $\varepsilon \leq \frac{1}{2}$ and we maintain weights w_e^t for all e that evolve as follows

$$w_e^0 = 1, \tag{1}$$

$$w_e^t = w_e^{t-1}\left(1+\frac{\varepsilon}{\rho}|f_e^t|\right), \tag{2}$$

where f^t is such that

$$\sum_e w_e^{t-1}|f_e^t| \leq \sum_e w_e^{t-1}, \tag{3}$$

for each t. (For simplicity, we assumed here that the worst-case overflow in each flow f^t is always ρ.)

Let us define our potential $\mu_t := \sum_e w_e^t$. Observe that $\mu_0 = m$.

Lemma 1 *For each $t \geq 1$,*

$$\mu_t \leq \mu_{t-1}\exp\left(\frac{\varepsilon}{\rho}\right).$$

In particular, $\mu_t \leq m\exp\left(\frac{\varepsilon}{\rho}\cdot t\right)$.

Proof Note that by (2) and the condition (3)

$$\mu_t = \sum_e w_e^t = \sum_e w_e^{t-1}\left(1+\frac{\varepsilon}{\rho}|f_e^t|\right) = \sum_e w_e^{t-1} + \frac{\varepsilon}{\rho}\sum_e w_e^{t-1}|f_e^t| \leq \sum_e w_e^{t-1}\left(1+\frac{\varepsilon}{\rho}\right) \leq \mu_{t-1}\exp\left(\frac{\varepsilon}{\rho}\right),$$

where the last inequality follows as $(1+x) \leq e^x$, for all $x \geq 0$.

Combining these inequalities for all $t' \leq t$ and using the fact that $\mu_0 = m$ gives us that also

$$\mu_t \leq m\exp\left(\frac{\varepsilon}{\rho}\cdot t\right),$$

as desired. ■

Lemma 2 *For any edge e,*

$$w_e^t \geq \exp\left((1-\varepsilon)\frac{\varepsilon}{\rho}\sum_{t'\leq t}|f_e^{t'}|\right).$$

Proof By (2),

$$w_e^t = w_e^{t-1}\left(1+\frac{\varepsilon}{\rho}|f_e^t|\right) \geq w_e^{t-1}\exp\left((1-\varepsilon)\frac{\varepsilon}{\rho}|f_e^t|\right),$$

where we used the fact that $(1+x) \geq \exp((1-x)x)$, whenever $0 \leq x \leq \frac{1}{2}$. (Observe that by definition of ρ, $\frac{\varepsilon}{\rho}|f_e^t| \leq \varepsilon \leq \frac{1}{2}$, so we can indeed apply this fact.[1])
Again, combining the inequality proved above for all $t' \leq t$ and recalling that $w_e^0 = 1$, gives us the lemma. ■

[1]This is the crucial (and only) place where our normalization of the multiplicative update by ρ plays role.

Now, to conclude the analysis of the multiplicative-weights update method (which we already did in the lecture), we note that, for any fixed edge e, trivially

$$w_e^N \leq \mu_N,$$

where N is the number of iterations of our multiplicative-weights update routine.
Using Lemmas 1 and 2, we obtain

$$\exp\left((1-\varepsilon)\frac{\varepsilon}{\rho}\sum_{t'\leq N}|f_e^{t'}|\right) \leq w_e^N \leq \mu_N \leq m\exp\left(\frac{\varepsilon}{\rho}\cdot N\right).$$

Taking a logarithm of both side and multiplying them by $\frac{\rho}{(1-\varepsilon)\varepsilon N}$ gives us

$$\frac{\sum_{t'\leq N}|f_e^{t'}|}{N} \leq \frac{\rho\log m}{(1-\varepsilon)\varepsilon N} + \frac{1}{1-\varepsilon}.$$

Observe that the amount of flow on the edge e in our final solution $\bar{f} := \frac{\sum_{t\leq N} f^t}{N}$ can be bounded by

$$|\bar{f}_e| = \left|\frac{\sum_{t\leq N} f_e^t}{N}\right| \leq \frac{\sum_{t\leq N}|f_e^t|}{N}.$$

So, combining two above inequalities we can see that whenever

$$N \geq \frac{\rho\log m}{(1-\varepsilon)\varepsilon^2}$$

we have that

$$|\bar{f}_e| \leq \frac{\rho\log m}{(1-\varepsilon)\varepsilon N} + \frac{1}{1-\varepsilon} \leq 1 + O(\varepsilon),$$

which is what we wanted to show. (We also used here the fact that $\frac{1}{(1-x)} \leq 1 + O(x)$, whenever $x \leq \frac{1}{2}$.)