Advanced Course on the Foundations of Computer Science 2014

Exercise Set

Lecturer: Aleksander Mądry

Problem 1. Let L be a Laplacian of a graph $G = (V, E, w)$ and let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be its eigenvalues.

(a) Show that $\lambda_1 = 0$ and the all-ones vector $\vec{1} := (1, \ldots, 1)$ is the corresponding eigenvector.

(b) Prove that, for any $k \geq 2$, $\lambda_k = 0$ iff G has at least k connected components.

Hint: The fact that, for any vector $x \in \mathbb{R}^n$, $x^T L x = \sum_{e=(u,v)\in E} w_e (x_u - x_v)^2$ might be useful here.

Problem 2. We say that a graph $G = (V, E)$ is *bipartite* iff we can partition its vertices into two sets P and Q such that each edge $e \in E$ has one of its endpoints in P and another one in Q. (In other words, we have that $P \cap Q = \emptyset$, $P \cup Q = V$, and $E \subseteq P \times Q$.)

(a) Prove that for *any* d-regular graph G, we have that $\lambda_n \leq 2d$.

(b) Show that if a graph G is bipartite and d-regular then its Laplacian matrix has an eigenvalue equal to $2d$.
Note: As long as G is connected, the above is in fact an "if and only if" condition, but you do not need to prove it.

Problem 3. In the lecture we provided the following two definitions of an electrical flow. *(All the matrix definition used below can be found in the appendix.)*
One of them stems from the Ohm's law:

Definition 1 *Given a graph G, a source s, a sink t, and a vector of resistances $r \in \mathbb{R}^m_+$, the electrical flow f of value 1 is defined as*

$$f = R^{-1} B^T \phi,$$

with $\phi \in \mathbb{R}^n$ being a vector of vertex potentials such that

$$L\phi = \chi_{s,t}, \tag{1}$$

where L is a Laplacian of G with weights $w_e := r_e^{-1}$, for each edge e.

The other one views electrical flows as a result of energy minimization over the space of all s-t flows of value 1 in G:

Definition 2 *Given a graph G, a source s, a sink t, and a vector of resistances r, f is an* electrical flow *(of value 1) iff*

$$f = \operatorname*{argmin}_{g \in \mathbb{R}^m : Bg = \chi_{s,t}} \sum_e r_e g_e^2. \tag{2}$$

In this problem, we want to show that these two definitions are dual to each other.

(a) Prove that the Lagrangian dual of the optimization problem (2) is

$$\max_{\eta \in \mathbb{R}^n} 2\eta^T \chi_{s,t} - \eta^T L \eta. \tag{3}$$

(b) Show that solving problem (3) corresponds to finding vertex potentials ϕ that satisfy (1).
Hint: Consider optimality conditions for problem (3).

Problem 4. It is well-known that any s-t flow f can be decomposed into a collection of flow cycles and flow s-t-paths. Prove that if f is an electrical flow then there can be no flow cycles in such decomposition.

Problem 5. For two vertices u, v of G and resistances $r \in \mathbb{R}^m$, we define the *effective resistance* $R_{eff}(u, v)$ *between* u *and* v as

$$R_{eff}(u, v) := \chi_{u,v}^T L^\dagger \chi_{u,v},$$

where L is the Laplacian of G wrt weights $w_e := r_e^{-1}$ and $L^\dagger$ is the pseudo-inverse of L.[1]

(a) Show that $R_{eff}(u, v)$ is equal to the energy of the electrical flow of value 1 in G wrt resistances r_e with u being the source and v being the sink.
Hint: Use the fact established in Problem 3 that $L^\dagger \chi_{u,v}$ *is an optimal solution to the problem* (3).

(b) Prove that the effective resistance forms a metric. That is, argue that:

   (I) $R_{eff}(u, v) = 0$ iff $u = v$;

   (II) $R_{eff}(u, v) = R_{eff}(v, u)$, for any u, v;

   (III) $R_{eff}(u, v) \leq R_{eff}(u, z) + R_{eff}(z, v)$, for any u, v, z.

Appendix: Definitions

Let $G = (V, E, w)$ be a weighted undirected graph $G = (V, E, w)$ with $m = |E|$ edges and $n = |V|$ vertices. Let us impose some arbitrary orientation of its edges and let s (resp. t) be a *source* (resp. *sink*) vertex. We define:

(1) *Edge-vertex incidence matrix* $B \in \mathbb{R}^{n \times m}$ to be a matrix such that

$$B_{v,e} := \begin{cases} -1 & \text{if } e = (v, u), \\ 1 & \text{if } e = (u, v), \\ 0 & \text{otherwise.} \end{cases}$$

(2) *Laplacian matrix* $L \in \mathbb{R}^{n \times n}$ to be a matrix such that

$$L_{v,u} := \begin{cases} -w_e & \text{if } e = (v, u) \in E, \\ \sum_{e \in E(v)} w_e & \text{if } v = u, \\ 0 & \text{otherwise,} \end{cases}$$

where $E(v)$ is the set of edges of G that are incident to v.

(3) *Resistance matrix* $R \in \mathbb{R}^{m \times m}$ to be a matrix such that

$$R_{e,e'} := \begin{cases} r_e & \text{if } e = e', \\ 0 & \text{otherwise.} \end{cases}$$

(4) *Demand vector* $\chi_{s,t} \in \mathbb{R}^n$ to be a vector such that

$$(\chi_{s,t})_v := \begin{cases} -1 & \text{if } v = s, \\ 1 & \text{if } v = t, \\ 0 & \text{otherwise.} \end{cases}$$

[1] Recall that a pseudo-inverse $A^\dagger$ of a symmetric matrix A is any matrix such that, for any x, $AA^\dagger x = x$ and $(A^\dagger Ax - x) \in ker(A)$.