ADFOCS 2014

Algorithmic challenges of Big Data for Optimization

Exercises, Wednesday, August 13, 2014

1. For *primal norm* $\|x\|$, $x \in R^n$, the *dual norm* is defined as $\|g\|_* = \max\limits_x \{\langle s, x\rangle : \ \|x\| \leq 1\}$. Compute the dual norm for the standard l_p-norms $\|x\|_p = \left[\sum\limits_{i=1}^n |x^{(i)}|^p \right]^{1/p}$ with $p \in [1, \infty]$. Why other values of p are not interesting?

2. Differentiable function $f(x)$ is called convex on R^n, if

$$f(y) \quad \geq \quad f(x) + \langle \nabla f(x), y - x \rangle, \quad x, y \in R^n.$$

It has Lipschitz continuous gradient if $\|\nabla f(x) - f(y)\|_* \leq L\|x - y\|$, $x, y \in R^n$. Prove two inequalities:

$$f(y) \quad \leq \quad f(x) + \langle \nabla f(x), y - x \rangle + \tfrac{1}{2} L \|y - x\|^2, \quad x, y \in R^n, \tag{1}$$

$$f(y) \quad \geq \quad f(x) + \langle \nabla f(x), y - x \rangle + \tfrac{1}{2L} \|\nabla f(y) - \nabla f(x)\|_*^2, \quad x, y \in R^n, \tag{2}$$

Can we say that (2) is a necessary and sufficient condition for convex functions, which have Lipschitz-continuous gradient with constant L?

3. Function $f(x)$ is called *strongly convex* with respect to the norm $\|\cdot\|$ if

$$f(y) \quad \geq \quad f(x) + \langle \nabla f(x), y - x \rangle + \tfrac{1}{2}\sigma \|y - x\|^2, \quad x, y \in R^n,$$

where the convexity parameter $\sigma > 0$. Prove that the problem of minimizing a strongly convex function on convex set always has a unique solution.

4. Prove that twice differentiable convex function is strongly convex on R^n if and only if

$$\langle \nabla f(x) h, h \rangle \quad \geq \quad \sigma \|h\|^2, \quad x, h \in R^n. \tag{3}$$

What is the value of convexity parameter of function $d(x) = \frac{1}{2}\|x\|_2^2$?

5. Prove that a positive definite matrix G satisfies inequality $\langle Gh, h \rangle \geq \sigma \|h\|^2$ for all $h \in R^n$ if and only if

$$\langle G^{-1} s, s \rangle \quad \leq \quad \tfrac{1}{\sigma} \|s\|_*^2, \quad s \in R^n. \tag{4}$$

6. Prove that strongly convex function f satisfies inequality

$$f(y) \quad \leq \quad f(x) + \langle \nabla f(x), y - x \rangle + \tfrac{1}{2\sigma} \|\nabla f(x) - \nabla f(y)\|_*^2, \quad x, y \in R^n. \tag{5}$$