ADFOCS 2014

Algorithmic challenges of Big Data for Optimization

Exercises, Thursday, August 14, 2014

1. For convex set $Q \subset R^n$, *Euclidean projection* $\pi_Q(x)$ of point $x \in R^n$ on Q is defined as follows:

$$\pi_Q(x) \quad = \quad \arg\min_{y \in Q} \|x - y\|_2.$$

Why it is unique? Find Euclidean projections for the following sets:

$$B_2 = \{y \in R^n : \|y\|_2 \leq 1, \quad B_\infty = \{y \in R^n : \|y\|_\infty \leq 1 \quad \Delta_n = \{y \in R^n_+ : \sum_{i=1}^n y^{(i)} = 1\}.$$

2. Prove that the *entropy function* $\eta(x) = \sum\limits_{i=1}^n x^{(i)} \ln x^{(i)}$ is strongly convex on Δ_n in l_1-norm. What is its convexity parameter.

3. Compute the Bregman projection

$$\mathcal{B}_h(x, g) = \arg\min_{y \in \Delta_n} \{h\langle g, y - x\rangle + \beta(x, y)\}$$

for the Bregman distance $\beta(x, y) = \eta(y) - \eta(x) - \langle \nabla\eta(x), y - x\rangle$, $x, y \in \Delta_n$.

4. For convex cone K, the *dual cone* K^* is defined as $K^* = \{s \in R^n : \langle s, x\rangle \geq 0, \forall x \in K\}$. Compute the dual cones for the following cones:

$$K = R^n_+, \quad K = S^n_+ \stackrel{\mathrm{def}}{=} \{X \in R^{n\times n} : X = X^T \succeq 0\}, \quad K = L_n \stackrel{\mathrm{def}}{=} \{(x, \tau) : \tau \geq \|x\|_2\}$$

(positive orthant, cone of positive semidefinite matrices, Lorentz cone).

5. For *primal* Conic Linear Problem $f^* = \inf\limits_{x \in K} \{\langle c, x\rangle : Ax = b\}$, where $b \in R^m$, the *dual* problem looks as follows:

$$f_* = \sup_{y \in R^m, s \in K^*} \{\langle b, y\rangle : s + A^T y = c\}.$$

Prove that for any primal feasible solution x and dual feasible solution (s, y) we have

$$\langle c, x\rangle - \langle b, y\rangle \geq 0,$$

implying $f^* \geq f_*$. Prove that primal-dual feasible solution (x^*, s^*, y^*) is optimal if and only if

$$\langle s^*, x^*\rangle = 0.$$