1. For convex set $Q \subset \mathbb{R}^n$, Euclidean projection $\pi_Q(x)$ of point $x \in \mathbb{R}^n$ on $Q$ is defined as follows:

$$\pi_Q(x) = \arg \min_{y \in Q} \|x - y\|_2.$$ Why it is unique? Find Euclidean projections for the following sets:

$$B_2 = \{y \in \mathbb{R}^n : \|y\|_2 \leq 1\}, \quad B_\infty = \{y \in \mathbb{R}^n : \|y\|_\infty \leq 1\}, \quad \Delta_n = \{y \in \mathbb{R}^n_+ : \sum_{i=1}^n y^{(i)} = 1\}.$$

2. Prove that the entropy function $\eta(x) = \sum_{i=1}^n x^{(i)} \ln x^{(i)}$ is strongly convex on $\Delta_n$ in $l_1$-norm. What is its convexity parameter.

3. Compute the Bregman projection

$$B_h(x, g) = \arg \min_{y \in \Delta_n} \{h(g, y - x) + \beta(x, y)\}$$

for the Bregman distance $\beta(x, y) = \eta(y) - \eta(x) - \langle \nabla \eta(x), y - x \rangle$, $x, y \in \Delta_n$.

4. For convex cone $K$, the dual cone $K^*$ is defined as $K^* = \{s \in \mathbb{R}^n : \langle s, x \rangle \geq 0, \forall x \in K\}$. Compute the dual cones for the following cones:

$$K = \mathbb{R}^n_+ \quad \text{def} \quad K = \mathbb{S}_+^n \quad \text{def} \quad \{X \in \mathbb{R}_+^{n \times n} : X = X^T \succeq 0\}, \quad K = L_n \quad \text{def} \quad \{(x, \tau) : \tau \geq \|x\|_2\}$$

(positive orthant, cone of positive semidefinite matrices, Lorentz cone).

5. For primal Conic Linear Problem $f^* = \inf_{x \in K} \{\langle c, x \rangle : Ax = b\}$, where $b \in \mathbb{R}^m$, the dual problem looks as follows:

$$f_* = \sup_{y \in \mathbb{R}^m, s \in K^*} \{\langle b, y \rangle : s + AT^Ty = c\}.$$ Prove that for any primal feasible solution $x$ and dual feasible solution $(s, y)$ we have

$$\langle c, x \rangle - \langle b, y \rangle \geq 0,$$

implying $f^* \geq f_*$. Prove that primal-dual feasible solution $(x^*, s^*, y^*)$ is optimal if and only if

$$\langle s^*, x^* \rangle = 0.$$