s-t path TSP

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ADFOCS

The traveling salesman problem

Traveling Salesman Problem (TSP)
Input:

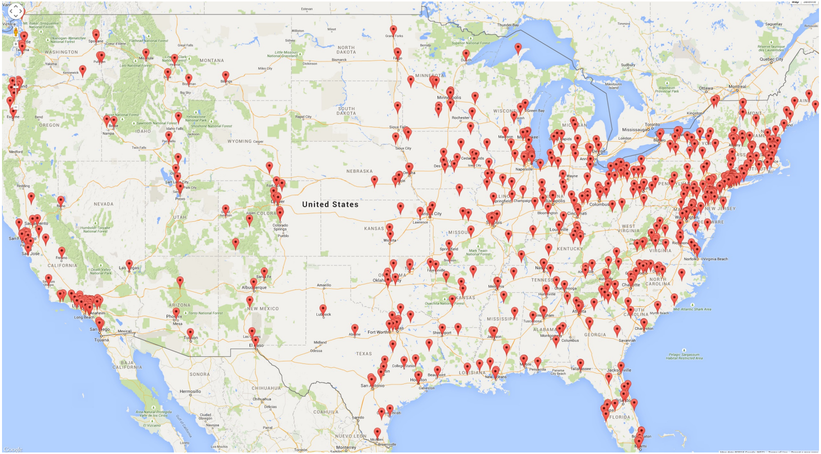
- A complete, undirected graph $G = (V, E)$;
- Edge costs $c(i,j) \geq 0$ for all $e = (i,j) \in E$.

Goal: Find the min-cost tour that visits each city exactly once.

Costs are *symmetric* ($c(i,j) = c(j,i)$) and obey the *triangle inequality* ($c(i,k) \leq c(i,j) + c(j,k)$).

Asymmetric TSP (ATSP) input has complete directed graph, and $c(i,j)$ may not equal $c(j,i)$.

The traveling salesman problem



From Bill Cook, tour of 647 US colleges
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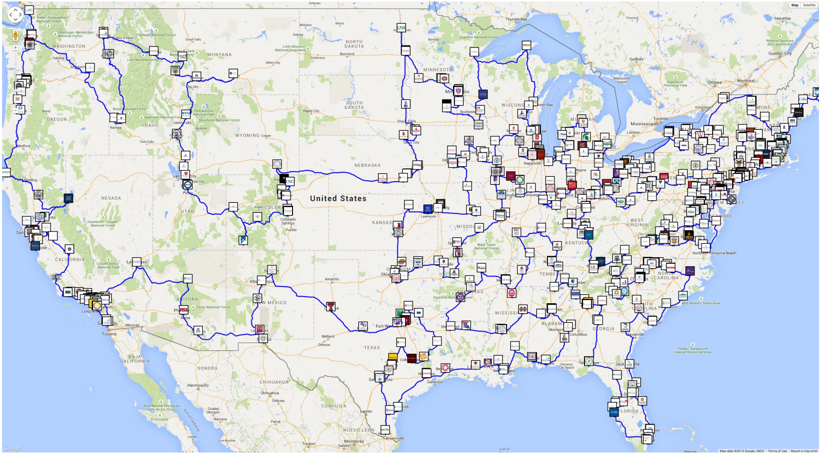

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Approximation Algorithms

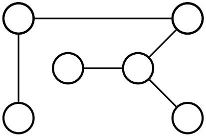
Definition

An α-approximation algorithm is a polynomial-time algorithm that returns a solution of cost at most α times the cost of an optimal solution.

Long known: A $\frac{3}{2}$-approximation algorithm due to Christofides (1976). No better approximation algorithm yet known.

Christofides' algorithm

Compute minimum spanning tree (MST) F on G, then compute a minimum-cost perfect matching M on odd-degree vertices of T. "Shortcut" Eulerian traversal in resulting Eulerian graph of $F \cup M$.



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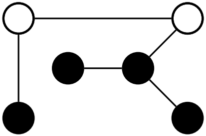

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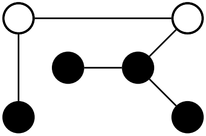

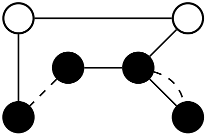

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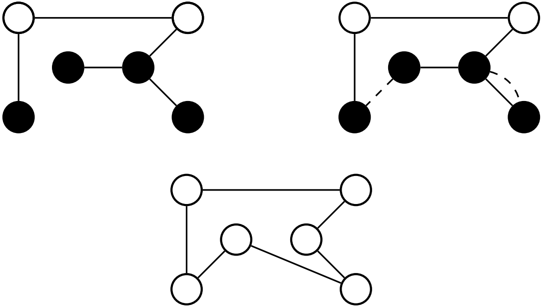

s-t path TSP

The s-t path TSP:
Usual TSP input plus $s, t \in V$, find a min-cost path from s to t visiting all other nodes in between (an *s-t Hamiltonian path*).

Hoogeveen (1991) shows that the natural variant of Christofides' algorithm gives a $\frac{5}{3}$-approximation algorithm.

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What is the natural variant for the s-t path TSP?

Eulerian path

There is an Eulerian path that starts at s, ends at t, and visits every edge exactly once iff s and t have odd-degree and all other vertices have even degree.

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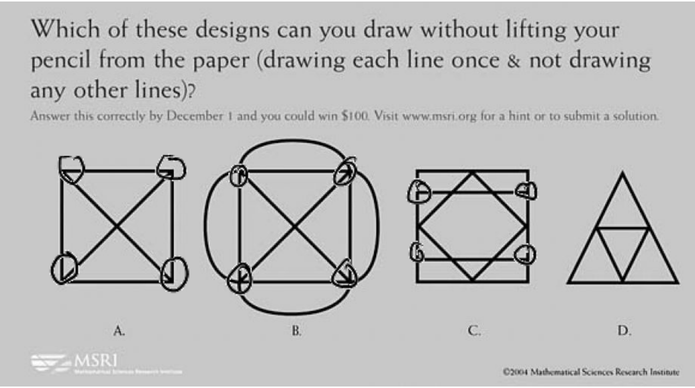
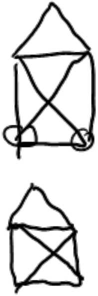
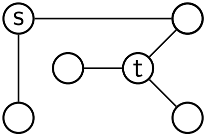




Hoogeveen's algorithm

Let F be the min-cost spanning tree. Let T be the set of vertices whose *parity needs changing*: s iff s has even degree in F, t iff t has even degree in F, and $v \neq s, t$ iff v has odd degree. Then find a minimum-cost perfect matching M on the vertices in T. Find Eulerian path on $F \cup M$; shortcut to an s-t Hamiltonian path.



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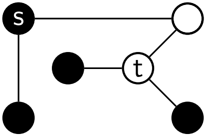

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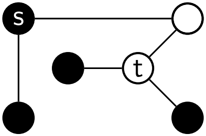

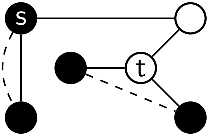

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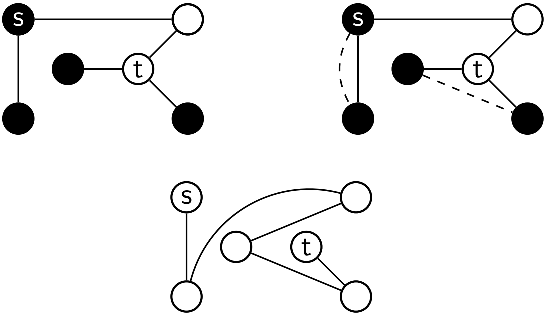
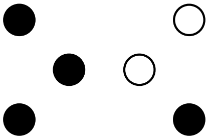


T-joins

Rather than a minimum-cost perfect matching on T, will construct a minimum-cost *T-join*: a set of edges that has odd degree at every vertex in T, even degree at every vertex not in T.



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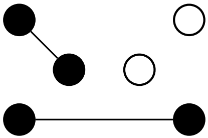

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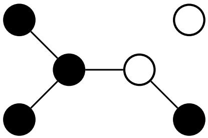

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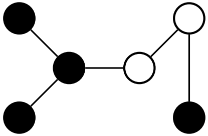

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Theorem

Hoogeveen's algorithm is a $\frac{5}{3}$-approximation algorithm.

Proof of theorem

Let F be edges in MST, $c(F) \equiv \sum_{e \in F} c_e$.

Let O be edges in optimal soln, $OPT = c(O)$.

Clearly $c(F) \leq OPT$ since O is a spanning tree.

Let T be vertices in F whose parity needs changing.

Idea: Construct 3 T-joins of total cost $c(F) + OPT$.

Then MST + min-cost T-join $\leq c(F) + \frac{1}{3}(c(F) + OPT)$

$\leq OPT + \frac{2}{3} OPT = \frac{5}{3} OPT$.

Let R be edges on s-t path in MST F.

Color edges of O green or blue: start at s, color blue until first node in T, then switch colors as each node in T reached. Gives G (green), B (blue).

$F-R$ a T-join: $F \cup (F-R)$ has even degree at every node except s, t

G a T-join: pairs up nodes in T.

B is not a T-join: $F \cup B$ has even degree at all nodes

But then $B \cup R$ is a T-join.

$$c(F-R) + c(G) + c(B \cup R) = c(F) + c(O).$$

$\blacksquare$

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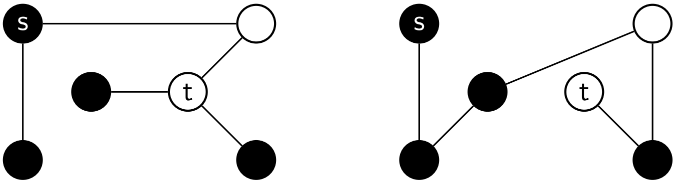

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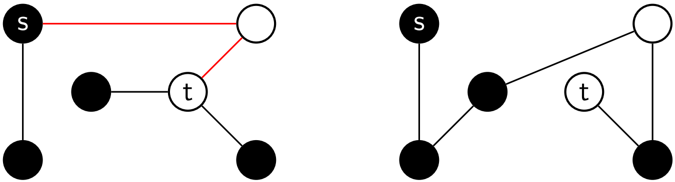

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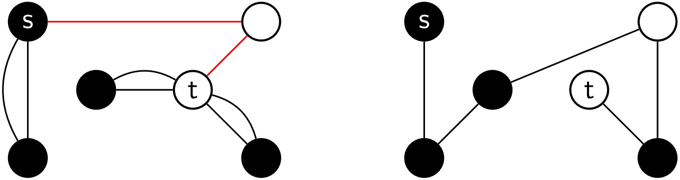

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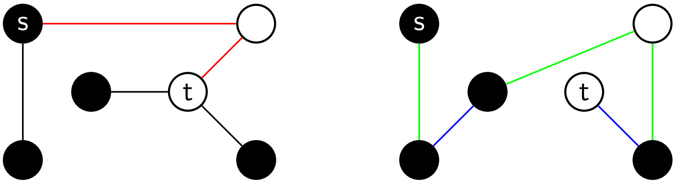

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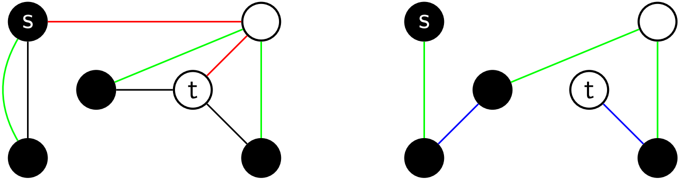

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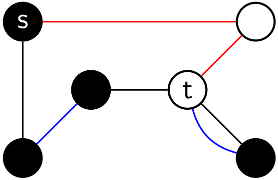

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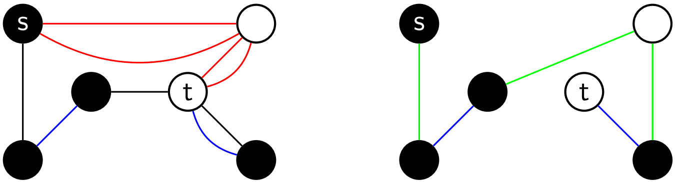
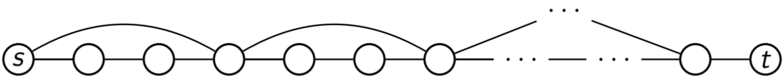


Tight Example

The analysis is tight. Consider the *graph TSP* instance below: cost c_e for $e = (i, j)$ is number of edges in shortest i-j path in graph.



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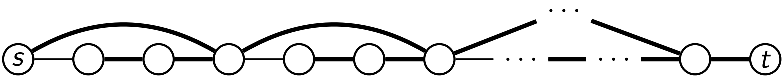

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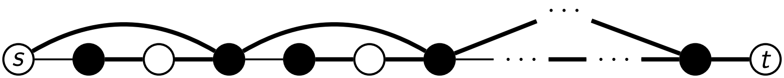

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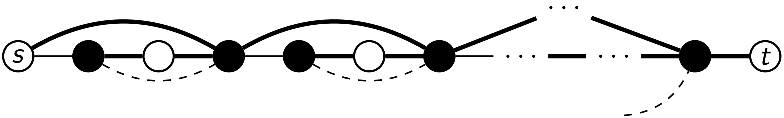

Improvements

No improvement on Hoogeveen's algorithm for s-t path TSP, until just the last few years.

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Vygen	(2015)	1.599

Goal: Understand the An et al. and Sebő algorithm and analysis.

A Linear Programming Relaxation

$$\text{Min} \quad \sum_{e \in E} c_e x_e$$

subject to:

$$x(\delta(v)) = \begin{cases} 1 & v = s,t \\ 2 & v \neq s,t \end{cases}$$

$$x(\delta(S)) \geq \begin{cases} 1 & |S \cap \{s,t\}| = 1 \\ 2 & |S \cap \{s,t\}| \neq 1 \end{cases}$$

$$0 \leq x_e \leq 1, \qquad \forall e \in E,$$

where $\delta(S)$ is the set of edges with exactly one endpoint in S, and $x(E') \equiv \sum_{e \in E'} x_e$.

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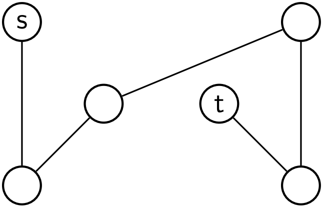
LP relaxation



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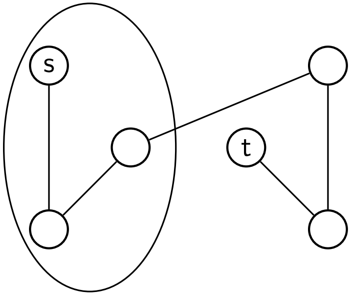

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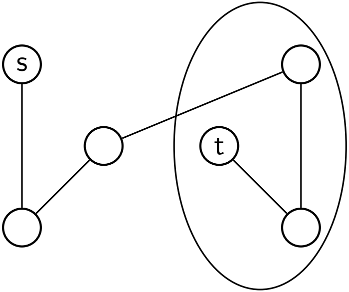

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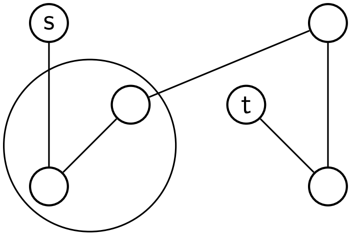

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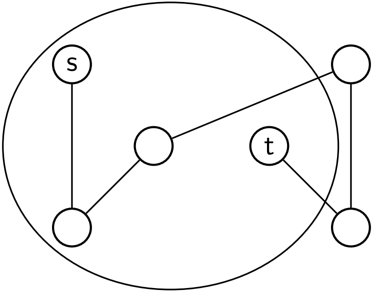
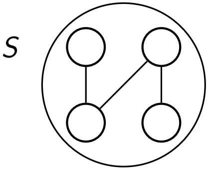


The spanning tree polytope

The spanning tree polytope (convex hull of all spanning trees) is defined by the following inequalities:

$$\begin{aligned} \sum_{e \in E} x_e = x(E) &= |V| - 1, & & \\ \sum_{e \in E(S)} x_e = x(E(S)) &\leq |S| - 1, & \forall |S| \subseteq V, |S| \geq 2, \\ x_e &\geq 0, & \forall e \in E, \end{aligned}$$

where $E(S)$ is the set of all edges with both endpoints in S.



The LP relaxation and spanning trees

Lemma

Any solution x feasible for the s-t path TSP LP relaxation is in the spanning tree polytope.

$$x(E) = \sum_{e \in E} x_e = \frac{1}{2} \sum_{v \in V} x(\delta(v))$$

$$= \frac{1}{2}\left((|V|-2)\cdot 2 + 2\right) = |V|-1$$

$$x(E(S)) = \frac{1}{2}\left(\sum_{v \in S} x(\delta(v)) - x(\delta(S))\right)$$

If $|S \cap \{s,t\}| = 1$

$$x(E(S)) \leq \frac{1}{2}\left(1 + 2(|S|-1) - 1\right) = |S|-1.$$

If $S \cap \{s,t\} = \emptyset$

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Proof

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A warmup to the improvements

Let OPT_{LP} be the value of an optimal solution x^* to the LP relaxation.

Theorem (An, Kleinberg, Shmoys (2012))

Hoogeveen's algorithm returns a solution of cost at most $\frac{5}{3} OPT_{LP}$.

An extremely useful lemma

Let F be a spanning tree, and let T be the vertices whose parity needs fixing in F.

Definition

S is an *odd set* if $|S \cap T|$ is odd.

Lemma

Let S be an odd set. If $|S \cap \{s,t\}| = 1$, then $|F \cap \delta(S)|$ is even. If $|S \cap \{s,t\}| \neq 1$, then $|F \cap \delta(S)|$ is odd.

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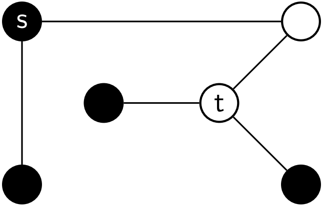

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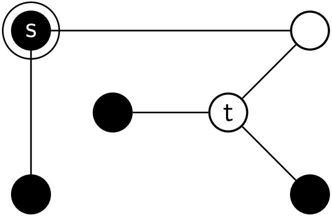

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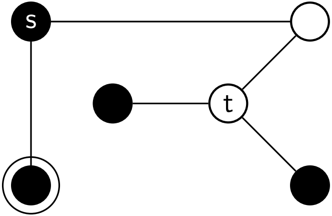

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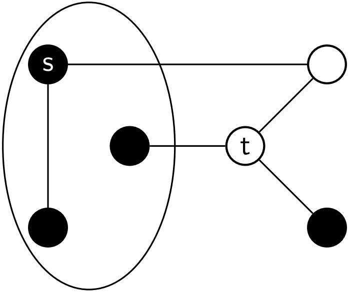
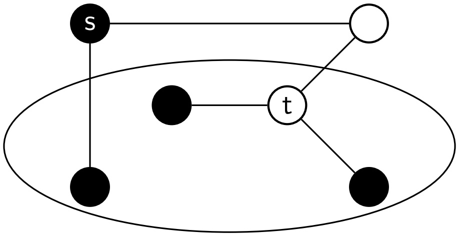
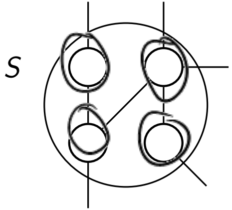


Lemma

Let S be an odd set. If $|S \cap \{s,t\}| = 1$, then $|F \cap \delta(S)|$ is even. If $|S \cap \{s,t\}| \neq 1$, then $|F \cap \delta(S)|$ is odd.



Proof of lemma



$$\sum_{v \in S} deg_F(v) = 2|E(S) \cap F| + |\delta(S) \cap F|$$

Pf of lemma

If $|S\cap\{s,t\}|=1$. Spse $s\in S$. $s\in T$ iff $\deg_F(s)$ even.

$\therefore$ S odd $\Rightarrow$ even # of odd deg. vertices in S.

$\therefore$ $|S\cap T|$ odd

$$\underbrace{\sum_{v\in S}\deg_F(v)}_{\text{even}} - \underbrace{2|E(S)\cap F|}_{\text{even}} = \underbrace{|\delta(S)\cap F|}_{\text{even}}$$

In fact $\underline{|\delta(S)\cap F|\geq 2}$

$|S\cap\{s,t\}|\neq 1$ & S odd $\Rightarrow$ odd # odd deg verts in S

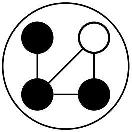
$$\underbrace{\sum\deg_F(v)}_{\text{odd}} - \underbrace{2|E(S)\cap F|}_{\text{even}} = \underbrace{|\delta(S)\cap F|}_{\text{odd}}$$

T-join LP

The solution to the following linear program is the minimum-cost T-join for costs $c \geq 0$:

$$\begin{array}{lll} & \text{Min} \quad \displaystyle\sum_{e \in E} c_e x_e & \\ \text{subject to:} & x(\delta(S)) \geq 1, & \forall S \subseteq V, |S \cap T| \text{ odd} \\ & x_e \geq 0, & \forall e \in E. \end{array}$$

S



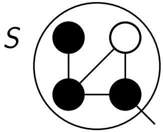
For $|S \cap T|$ odd

$$\underbrace{\sum_{v \in S} deg_J(v)}_{\text{odd}} = \underbrace{2|E(S) \cap J|}_{\text{even}} + \underbrace{|\delta(S) \cap J|}_{\text{odd}}$$

T-join LP

The solution to the following linear program is the minimum-cost T-join for costs $c \geq 0$:

$$\begin{array}{llll} & \text{Min} & \sum_{e \in E} c_e x_e & \\ \text{subject to:} & & x(\delta(S)) \geq 1, & \forall S \subseteq V, |S \cap T| \text{ odd} \\ & & x_e \geq 0, & \forall e \in E. \end{array}$$



$$\sum_{v \in S} deg_J(v) = 2|E(S) \cap J| + |\delta(S) \cap J|$$

Proof of theorem

Theorem (An, Kleinberg, Shmoys (2012))

Hoogeveen's algorithm returns a solution of cost at most $\frac{5}{3} OPT_{LP}$.

Lemma

Let S *be an odd set. If* $|S \cap \{s,t\}| = 1$, *then* $|F \cap \delta(S)|$ *is even. If* $|S \cap \{s,t\}| \neq 1$, *then* $|F \cap \delta(S)|$ *is odd.*

$$\begin{array}{lll} \text{Min} & \displaystyle\sum_{e \in E} c_e x_e & \\ & x(\delta(S)) \geq 1, & \forall S \subseteq V, |S \cap T| \text{ odd} \\ & x_e \geq 0, & \forall e \in E. \end{array}$$

Pf of thm: Let x^* be an opt. soln to LP relaxation.

Cost of MST $\leq \sum_{e \in E} c_e x_e^* = OPT_{LP}$.

since x^* is feasible for spanning tree polytope.

Let $\chi_F \in \{0,1\}^{|E|}$ s.t. $\chi_F(e) = \begin{cases} 1 & \text{if } e \in F \\ 0 & \text{o.w.} \end{cases}$

Claim: $y = \frac{1}{3}\chi_F + \frac{1}{3}x^*$ feasible for T-join LP.

Then $c(F \cup J) = c(F) + c(J) \leq OPT_{LP} + \frac{1}{3}c(F) + \frac{1}{3}OPT_{LP}$

$\leq \frac{5}{3} OPT_{LP}$

$y = \frac{1}{3}\chi_F + \frac{1}{3}x^*$ feas. for T-join LP.

Need to show that if $|S \cap T|$ odd, then $y(\delta(S)) \geq 1$.

If $|S \cap \{s,t\}| \neq 1$, then

$$y(\delta(S)) = \frac{1}{3}|F \cap \delta(S)| + \frac{1}{3}x^*(\delta(S)) \geq \frac{1}{3} + \frac{2}{3} = 1$$

If $|S \cap \{s,t\}| = 1$, then

$$y(\delta(S)) = \frac{1}{3}\underbrace{|F \cap \delta(S)|}_{\geq 2} + \frac{1}{3}x^*(\delta(S)) \geq \frac{2}{3} + \frac{1}{3} = 1$$

∎

Convex combination

Let x^* be an optimal LP solution. Let χ_F be the *characteristic vector* of a set of edges F, so that

$$\chi_F(e) = \begin{cases} 1 & e \in F \\ 0 & e \notin F \end{cases}$$

Since x^* is in the spanning tree polytope, can write x^* as a convex combination of spanning trees $F_1, \ldots, F_k$:

$$x^* = \sum_{i=1}^{k} \lambda_i \chi_{F_i},$$

such that $\sum_{i=1}^{k} \lambda_i = 1$, $\lambda_i \geq 0$.

Best-of-Many Christofides' Algorithm

An, Kleinberg, Shmoys (2012) propose the *Best-of-Many Christofides'* algorithm: given optimal LP solution x^*, compute convex combination of spanning trees

$$x^* = \sum_{i=1}^{k} \lambda_i \chi_{F_i}.$$

For each spanning tree F_i, let T_i be the set of vertices whose parity needs fixing, let J_i be the minimum-cost T_i-join. Find s-t Hamiltonian path by shortcutting $F_i \cup J_i$. Return the shortest path found over all i.

Best-of-Many Christofides' Algorithm

$$x^* = \sum_{i=1}^{k} \lambda_i \chi_{F_i}.$$

For each spanning tree F_i, let T_i be the set of vertices whose parity needs fixing, J_i be the minimum-cost T_i-join. Find s-t Hamiltonian path by shortcutting $F_i \cup J_i$. Return the shortest path found over all i.

Theorem

The Best-of-Many Christofides' algorithm returns a solution of cost at most $\frac{5}{3}OPT_{LP}$.