s-t path TSP

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ADFOCS

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The s-t path traveling salesman problem

The s-t Path Traveling Salesman Problem (s-t Path TSP)

Input:

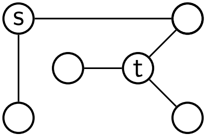
- A complete, undirected graph $G = (V, E)$;
- Edge costs $c(i, j) \geq 0$ for all $e = (i, j) \in E$;
- Vertices $s, t \in V$.

Goal: Find the min-cost path that starts at s, ends at t, and visits every other vertex exactly once.

Costs are *symmetric* ($c(i, j) = c(j, i)$) and obey the *triangle inequality* ($c(i, k) \leq c(i, j) + c(j, k)$).

Hoogeveen's algorithm

Let F be the min-cost spanning tree. Let T be the set of vertices whose *parity needs changing*: s iff s has even degree in F, t iff t has even degree in F, and $v \neq s, t$ iff v has odd degree. Then find a minimum-cost T-join J. Find Eulerian path on $F \cup J$; shortcut to an s-t Hamiltonian path.



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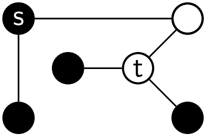

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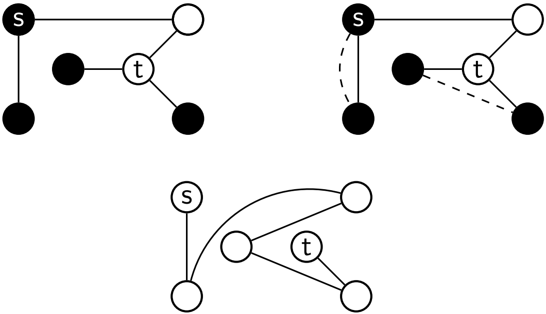


Improvements

Course 1

Theorem

Hoogeveen's algorithm is a $\frac{5}{3}$-approximation algorithm.

Recent improvements on Hoogeveen's algorithm.

Hoogeveen	(1991)	$\frac{5}{3}$
An, Kleinberg, Shmoys	(2012)	$\frac{1+\sqrt{5}}{2} \approx 1.618$
Sebő	(2013)	$\frac{8}{5} = 1.6$
Vygen	(2015)	1.599

Goal: Understand the An et al. and Sebő algorithm and analysis.

A Linear Programming Relaxation

$$\text{Min} \quad \sum_{e \in E} c_e x_e$$

subject to:

$$x(\delta(v)) = \begin{cases} 1, & v = s, t, \\ 2, & v \neq s, t, \end{cases}$$

$$x(\delta(S)) \geq \begin{cases} 1, & |S \cap \{s,t\}| = 1, \\ 2, & |S \cap \{s,t\}| \neq 1, \end{cases}$$

$$0 \leq x_e \leq 1, \qquad \forall e \in E,$$

where $\delta(S)$ is the set of edges with exactly one endpoint in S, and $x(E') \equiv \sum_{e \in E'} x_e$.

The spanning tree polytope

The spanning tree polytope (convex hull of all spanning trees) is defined by the following inequalities:

$$\begin{aligned} x(E) &= |V| - 1, & & \\ x(E(S)) &\leq |S| - 1, & & \forall |S| \subseteq V, |S| \geq 2, \\ x_e &\geq 0, & & \forall e \in E, \end{aligned}$$

where $E(S)$ is the set of all edges with both endpoints in S.

Lemma

Any solution x feasible for the s-t path TSP LP relaxation is in the spanning tree polytope.

Course

A warmup to the improvements

Let OPT_{LP} be the value of an optimal solution x^* to the LP relaxation.

Theorem (An, Kleinberg, Shmoys (2012))

Hoogeveen's algorithm returns a solution of cost at most $\frac{5}{3}OPT_{LP}$.

An extremely useful lemma

Let F be a spanning tree, and let T be the vertices whose parity needs fixing in F.

Definition

S is an *odd set* if $|S \cap T|$ is odd.

Lemma

Let S be an odd set. If $|S \cap \{s,t\}| = 1$, then $|F \cap \delta(S)|$ is even. If $|S \cap \{s,t\}| \neq 1$, then $|F \cap \delta(S)|$ is odd.

$|F \cap \delta(S)| \geq 2$

If S odd, $|S \cap \{s,t\}| = 1$

T-join LP

The solution to the following linear program is the minimum-cost T-join for costs $c \geq 0$:

$$
\begin{array}{lll}
\text{Min} & \displaystyle\sum_{e\in E} c_e x_e & \\
\text{subject to:} & x(\delta(S)) \geq 1, & \forall S \subseteq V, |S \cap T| \text{ odd} \\
& x_e \geq 0, & \forall e \in E.
\end{array}
$$

Proof of theorem

Theorem (An, Kleinberg, Shmoys (2012))

Hoogeveen's algorithm returns a solution of cost at most $\frac{5}{3}OPT_{LP}$.

Lemma

Let S *be an odd set. If* $|S \cap \{s,t\}| = 1$, *then* $|F \cap \delta(S)|$ *is even. If* $|S \cap \{s,t\}| \neq 1$, *then* $|F \cap \delta(S)|$ *is odd.*

$$\begin{aligned} \text{Min} \quad & \sum_{e \in E} c_e x_e \\ & x(\delta(S)) \geq 1, && \forall S \subseteq V, |S \cap T| \text{ odd} \\ & x_e \geq 0, && \forall e \in E. \end{aligned}$$

Basic idea: Show that $y = \frac{1}{3}\chi_F + \frac{1}{3}x^*$ is feasible for T-join LP, where x^* is solution to LP relaxation, and χ_F is characteristic vector for spanning tree F.

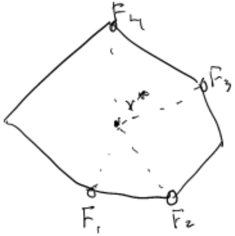
Convex combination

Let x^* be an optimal LP solution. Let χ_F be the *characteristic vector* of a set of edges F, so that

$$\chi_F(e) = \begin{cases} 1 & e \in F \\ 0 & e \notin F \end{cases}$$

Since x^* is in the spanning tree polytope, can write x^* as a convex combination of spanning trees $F_1, \ldots, F_k$:

$$x^* = \sum_{i=1}^{k} \lambda_i \chi_{F_i},$$



such that $\sum_{i=1}^{k} \lambda_i = 1$, $\lambda_i \geq 0$.

Best-of-Many Christofides' Algorithm

An, Kleinberg, Shmoys (2012) propose the *Best-of-Many Christofides'* algorithm: given optimal LP solution x^*, compute convex combination of spanning trees

$$x^* = \sum_{i=1}^{k} \lambda_i \chi_{F_i}.$$

For each spanning tree F_i, let T_i be the set of vertices whose parity needs fixing, let J_i be the minimum-cost T_i-join. Find s-t Hamiltonian path by shortcutting $F_i \cup J_i$. Return the shortest path found over all i.

Best-of-Many Christofides' Algorithm

$$x^* = \sum_{i=1}^{k} \lambda_i \chi_{F_i}.$$

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Theorem

The Best-of-Many Christofides' algorithm returns a solution of cost at most $\frac{5}{3} OPT_{LP}$.

Proof Course 3

$y_i = \frac{1}{3}\chi_{F_i} + \frac{1}{3}x^*$. Claim: Feasible for T_i-join LP.

If S odd, $|S \cap \{s,t\}| \neq 1$, then

$$y_i(\delta(S)) = \frac{1}{3}|F_i \cap \delta(S)| + \frac{1}{3}x^*(\delta(S)) \geq \frac{1}{3} + \frac{2}{3}$$

If S odd, $|S \cap \{s,t\}| = 1$, then

$$y_i(\delta(S)) = \frac{1}{3}|F_i \cap \delta(S)| + \frac{1}{3}x^*(\delta(S)) \geq \frac{2}{3} + \frac{1}{3} = 1$$

Cost of $\min_i c(F_i \cup J_i) \leq \sum_i \lambda_i c(F_i \cup J_i)$

$$= \sum_i \lambda_i \left(\sum_{e \in F_i} c_e + \frac{1}{3}\sum_{e \in F_i} c_e + \frac{1}{3}\sum_{e \in E} c_e x_e^* \right)$$

$$= \sum_i \lambda_i \left(\frac{4}{3} c(F_i) + \frac{1}{3}\sum_{e \in E} c_e x_e^* \right)$$

$$= \frac{4}{3}\sum_i \lambda_i c(F_i) + \frac{1}{3}\sum_i \lambda_i \sum_{e \in E} c_e x_e^*$$

$$\leq \frac{4}{3}\sum_{e \in E} c_e x_e^* + \frac{1}{3}\sum_{e \in E} c_e x_e^* = \frac{5}{3}\sum_{e \in E} c_e x_e^* = \frac{5}{3}OPT_{LP}$$

Improvement?

To do better, we need to improve the analysis for the costs of the T_i-joins; recall that we use that

$$y_i = \frac{1}{3}\chi_{F_i} + \frac{1}{3}x^*$$

is feasible for the T_i-join LP.

Consider

$$y_i = \alpha\chi_{F_i} + \beta x^*.$$

If feasible for T_i-join LP

Then the cost of the best s-t Hamiltonian path is at most

$$(1 + \alpha + \beta)OPT_{LP}.$$

Improvement?

Proof that y_i feasible for T_i-join LP had two cases. Assume S odd ($|S \cap T_i|$ odd).

If $|S \cap \{s, t\}| \neq 1$, then

$$y_i(\delta(S)) = \alpha|\overset{\geq 1}{F_i \cap \delta(S)}| + \beta \overset{\geq 2}{x^*(\delta(S))} \geq \alpha + 2\beta.$$

We will want $\alpha + 2\beta \geq 1$, so the T_i-join LP constraint is satisfied.

Improvement?

If $|S \cap \{s,t\}| = 1$, then

$$y_i(\delta(S)) = \alpha\overset{\geq 2}{|F_i \cap \delta(S)|} + \beta x^*(\delta(S)) \geq 2\alpha + \beta x^*(\delta(S)).$$

Improvement?

If $|S \cap \{s, t\}| = 1$, then

$$y_i(\delta(S)) = \alpha|F_i \cap \delta(S)| + \beta x^*(\delta(S)) \geq \underbrace{2\alpha + \beta x^*(\delta(S))}_{<1}.$$

Since we assume $\alpha + 2\beta \geq 1$, we only run into problems if

$$x^*(\delta(S)) < \frac{1 - 2\alpha}{\beta}.$$

Note that $\alpha = 0$, $\beta = \frac{1}{2}$ works if $x^*(\delta(S)) \geq 2$ for all $S \subset V$, and gives a tour of cost at most $\frac{3}{2}OPT_{LP}$.

Improvement?

If $|S \cap \{s,t\}| = 1$, then

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So focus on cuts for which $x^*(\delta(S)) < 2$, and add an extra "correction" term to y_i to handle these cuts.

τ-Narrow Cuts

Definition

S is τ-*narrow* if $x^*(\delta(S)) < 1 + \tau$ for fixed $\tau \leq 1$.

Only S such that $|S \cap \{s, t\}| = 1$ are τ-narrow.

Definition

Let $\mathcal{C}_\tau$ be all τ-narrow cuts S with $s \in S$.

τ-Narrow Cuts

The τ-narrow cuts in $\mathcal{C}_\tau$ have a nice structure.

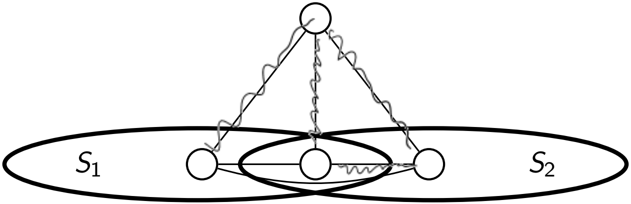
Course 1

Theorem (An, Kleinberg, Shmoys (2012))

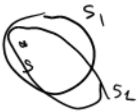
If $S_1, S_2 \in \mathcal{C}_\tau$, $S_1 \neq S_2$, then either $S_1 \subset S_2$ or $S_2 \subset S_1$.

First need to show that

$$x^*(\delta(S_1)) + x^*(\delta(S_2)) \geq x^*(\delta(S_1 - S_2)) + x^*(\delta(S_2 - S_1)).$$



Pf Sppse otherwise. $S_1 - S_2 \neq \emptyset$, and $S_2 - S_1 \neq \emptyset$



$$
\begin{aligned}
(1+\tau) + (1+\tau) &> x^*(\delta(S_1)) + x^*(\delta(S_2)) \\
&\geq x^*(\delta(S_1 - S_2)) + x^*(\delta(S_2 - S_1)) \\
&\geq 2 + 2. \qquad \rightarrow\leftarrow
\end{aligned}
$$

$\square$

Proof of theorem

Theorem (An, Kleinberg, Shmoys (2012))

If $S_1, S_2 \in \mathcal{C}_\tau$, $S_1 \neq S_2$, then either $S_1 \subset S_2$ or $S_2 \subset S_1$.

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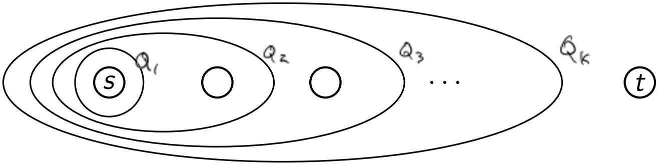
So the τ-narrow cuts look like $s \in Q_1 \subset Q_2 \subset \cdots \subset Q_k \subset V$.



Correction Factor

Let e_Q be the minimum-cost edge in $\delta(Q)$. Then consider the following (from Gao (2014)):

$$y_i = \alpha\chi_{F_i} + \beta x^* + \sum_{Q \in \mathcal{C}_\tau, |Q \cap T_i| \text{ odd}} (1 - 2\alpha - \beta x^*(\delta(Q)))\, \chi_{e_Q}$$

for $\alpha, \beta, \tau \geq 0$ such that

$$\alpha + 2\beta = 1 \qquad \text{and} \qquad \tau = \frac{1-2\alpha}{\beta} - 1.$$

Course 2

Theorem

y_i is feasible for the T_i-join LP.

Pf S odd ($|S \cap T_i|$ odd)

If $|S \cap \{s,t\}| \neq 1$

$y_i(\delta(S)) \geq \alpha |F_i \cap \delta(S)| + \beta x^*(\delta(S)) \geq \alpha + 2\beta = 1$ ✓

If $|S \cap \{s,t\}| = 1$

If S not τ-narrow

$y_i(\delta(S)) \geq \alpha |F_i \cap \delta(S)| + \beta x^*(\delta(S)) \geq 2\alpha + \beta(1+\tau) = 1.$ ✓

If S is τ-narrow

$$y_i(\delta(S)) \geq \alpha |F_i \cap \delta(S)| + \beta x^*(\delta(S)) + (1 - 2\alpha - \beta x^*(\delta(S)))|e_S \cap \delta(S)|$$

$$\geq 2\alpha + \beta x^*(\delta(S)) + (1 - 2\alpha - \beta x^*(\delta(S)))$$

$$= 1.$$

Proof

Two Lemmas

Recall $x^* = \sum_{i=1}^{k} \lambda_i \chi_{F_i}$, with $\sum_{i=1}^{k} \lambda_i = 1$ and $\lambda_i \geq 0$. So λ_i is a probability distribution on the trees F_i; probability of F_i is λ_i.

Lemma

Let $\mathcal{F}$ be a randomly sampled tree F_i, and $\mathcal{T}$ the corresponding vertices T_i. Let $Q \in \mathcal{C}_\tau$ be a τ-narrow cut. Then

$$\Pr[|\delta(Q) \cap \mathcal{F}| = 1] \geq 2 - x^*(\delta(Q))$$
$$\Pr[|Q \cap \mathcal{T}| \text{ odd}] \leq x^*(\delta(Q)) - 1.$$

$$x^*(\delta(Q)) = E[|\mathcal{J} \cap \delta(Q)|] \geq \Pr[|\mathcal{J} \cap \delta(Q)| = 1] + 2\Pr[|\mathcal{J} \cap \delta(Q)| \geq 2]$$

and $\Pr[|\mathcal{J} \cap \delta(Q)| = 1] + \Pr[|\mathcal{J} \cap \delta(Q)| \geq 2] = 1$

$$\therefore \Pr[|\mathcal{J} \cap \delta(Q)| = 1] \geq 2 - x^*(\delta(Q))$$

$$\Pr[|\mathcal{J} \cap \delta(Q)| \geq 2] \leq x^*(\delta(Q)) - 1$$

Recall $|Q \cap T_i|$ odd $\Rightarrow |F_i \cap \delta(Q)| \geq 2$

$$\therefore \Pr[|Q \cap T_i| \text{ odd}] \leq \Pr[|\mathcal{J} \cap \delta(Q)| \geq 2] \leq x^*(\delta(Q)) - 1.$$

$\square$

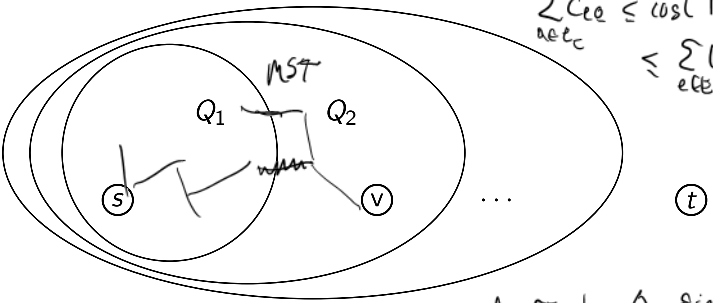
Two Lemmas

Recall e_Q is the cheapest edge crossing a τ-narrow cut $Q \in \mathcal{C}_\tau$.

Lemma

$$\sum_{Q \in \mathcal{C}_\tau} c_{e_Q} \le \sum_{e \in E} c_e x_e^*.$$

$$\sum_{Q \in \mathcal{C}_\tau} c_{e_Q} \le \text{cost MST} \le \sum_{e \in E} c_e x_e^*$$



Idea: For each $Q \in \mathcal{C}_\tau$, map a unique edge e of MST to Q. Since $c_{e_Q} \le c_e$.
To induct, remove e, contract s and v, repeat

An-Kleinberg-Shmoys

Course 4 - 5

Theorem (An, Kleinberg, and Shmoys (2012))

Best-of-Many Christofides' is a $\frac{1+\sqrt{5}}{2}$-approximation algorithm for s-t path TSP.

Pf Best s-t path $\leq \sum_i \lambda_i \, c(F_i \cup J_i)$

$$= \sum_i \lambda_i \left[c(F_i) + \alpha\, c(F_i) + \beta \sum_{e \in E} c_e x_e^* + \sum_{\substack{Q \in \mathcal{C}_\tau \\ |Q \cap T_i| \text{ odd}}} \left(1 - 2\alpha - \beta x^*(\delta(Q))\right) c_{e_Q} \right]$$

$$\leq (1+\alpha+\beta) \sum_{e \in E} c_e x_e^* + \sum_{Q \in \mathcal{C}_\tau} \left(x^*(\delta(Q)) - 1\right)\left(1 - 2\alpha - \beta x^*(\delta(Q))\right) c_{e_Q}$$

$$\leq ((1+\alpha+\beta) \sum_{e \in E} c_e x_e^* + \max_{0 \leq z < \tau} z\left(1 - 2\alpha + \beta(1+z)\right) \sum_{Q \in \mathcal{C}_\tau} c_{e_Q}$$

$$\leq \left((1+\alpha+\beta + \max_{0 \leq z < \tau} z\left(1 - 2\alpha - \beta(1+z)\right)\right) \sum_{e \in E} c_e x_e^*$$

$$= ((1+\alpha+\beta \max_{0 \leq z < \tau} z\left(\beta\tau - \beta z\right)) \sum_{e \in E} c_e x_e^*$$

Maximized at $z = \tau/2$

$$\leq \left((1+\alpha+\beta+\beta\left(\tfrac{\tau}{2}\right)^2\right) OPT_{LP}$$

$$\rightsquigarrow \leq \left(2 - \beta + \frac{(3\beta-1)^2}{4\beta}\right) OPT_{LP}$$

Proof of AKS

For the proof, recall that e_Q is min-cost edge in $\delta(Q)$, $\mathcal{C}_\tau$ are the cuts Q with $x^*(\delta(Q)) < 1+\tau$,

$$y_i = \alpha\chi_{F_i} + \beta x^* + \sum_{Q\in\mathcal{C}_\tau, |Q\cap T_i| \text{ odd}} (1-2\alpha-\beta x^*(\delta(Q)))\,\chi_{e_Q}$$

is feasible for the T_i-join LP, and

Lemma

Let $\mathcal{F}$ be a randomly sampled tree F_i, and $\mathcal{T}$ the corresponding vertices T_i. Let $Q \in \mathcal{C}_\tau$ be a τ-narrow cut. Then

$$\Pr[|\delta(Q)\cap\mathcal{F}| = 1] \geq 2 - x^*(\delta(Q))$$

$$\Pr[|Q\cap\mathcal{T}| \textit{ odd}] \leq x^*(\delta(Q)) - 1.$$

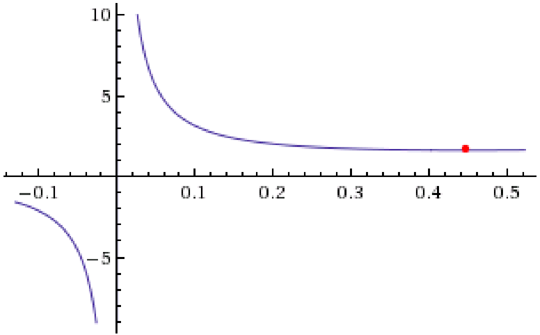
Lemma

$$\sum_{Q\in\mathcal{C}_\tau} c_{e_Q} \leq \sum_{e\in E} c_e x^*_e.$$

Global minimum:

$$\min\left\{2 - x + \frac{(3x-1)^2}{4x} \;\middle|\; x \geq 0\right\} = \frac{1}{2}\left(1+\sqrt{5}\right) \text{ at } x = \frac{1}{\sqrt{5}}$$

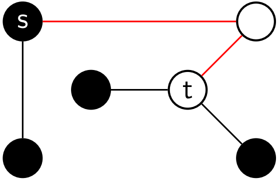
Plot:



(x from −0.1 to 0.5)

Sebő's Improvement

Sebő (2013) gives a tighter analysis of the Best-of-Many Christofides' algorithm. For spanning tree F_i, let F_i^{st} be the set of edges in the s-t path in F_i. Recall from the proof of Hoogeven's algorithm that $F_i - F_i^{st}$ is also a T_i-join, so $c(J_i) \leq c(F_i - F_i^{st})$.



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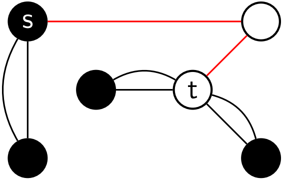

One More Lemma

Let $\mathcal{F}$ be a random spanning tree (tree F_i with probability λ_i), and $\mathcal{F}^{st}$ its associated s-t path. Let $c(\mathcal{F}^{st})$ be the cost of this path. Recall that

$$\Pr[|\mathcal{F} \cap \delta(Q)| = 1] \geq 2 - x^*(\delta(Q))$$

for a τ-narrow cut Q.

Lemma (Sebő (2013))

$$\sum_{Q \in \mathcal{C}_\tau} (2 - x^*(\delta(Q)))c_{e_Q} \leq E[c(\mathcal{F}^{st})].$$

Sebő (2013)

Theorem (Sebő (2013))

Best-of-Many Christofides' is an $\frac{8}{5}$-approximation algorithm.

Proof of Sebő

For the proof, recall that e_Q is min-cost edge in $\delta(Q)$, $\mathcal{C}_\tau$ are the cuts Q with $x^*(\delta(Q)) < 1+\tau$,

$$y_i = \alpha \chi_{F_i} + \beta x^* + \sum_{Q \in \mathcal{C}_\tau, |Q \cap T_i| \text{ odd}} (1 - 2\alpha - \beta x^*(\delta(Q)))\, \chi_{e_Q}$$

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Lemma

$$\sum_{Q \in \mathcal{C}_\tau} (2 - x^*(\delta(Q))) c_{e_Q} \leq E[c(\mathcal{F}^{st})].$$

maximize	function	$x \times \frac{3-4x}{9-9x}$
	domain	$0.75 \geq x \geq 0$

Global maximum:

$$\max\left\{\frac{x(3-4x)}{9-9x} \,\middle|\, 0.75 \geq x \geq 0\right\} \approx 0.111111 \text{ at } x \approx 0.5$$

Vygen's Improvement

Vygen (2015) gives a 1.599-approximation algorithm.

Vygen's Improvement

Vygen (2015) gives a 1.599-approximation algorithm.

Key idea: Modify the initial convex combination of trees into another one that avoids certain bad properties.

Integrality Gap

The performance of Best-of-Many Christofides' cannot do better than the *integrality gap* of the LP relaxation.

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The performance of Best-of-Many Christofides' cannot do better than the *integrality gap* of the LP relaxation.

The integrality gap is

$$\mu \equiv \sup \frac{OPT}{OPT_{LP}}$$

over all instances of the problem.

Integrality Gap

The performance of Best-of-Many Christofides' cannot do better than the *integrality gap* of the LP relaxation.

The integrality gap is

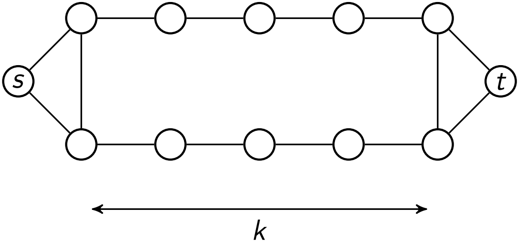
$$\mu \equiv \sup \frac{OPT}{OPT_{LP}}$$

over all instances of the problem.

Note that we have shown $\mu \leq \frac{8}{5}$, since we find a tour of cost at most $\frac{8}{5} OPT_{LP}$.

Integrality Gap

We can show a lower bound on the integrality gap using an instance of *graph TSP*: input is a graph $G = (V, E)$, cost c_e for $e = (i, j)$ is number of edges in a shortest i-j path in G.



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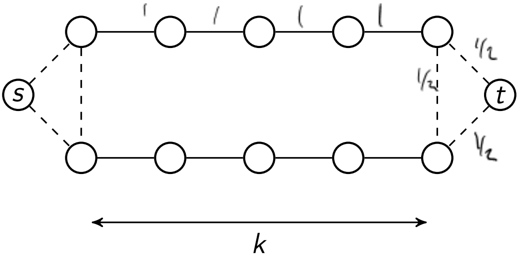

$$OPT_{LP} \approx 2k$$

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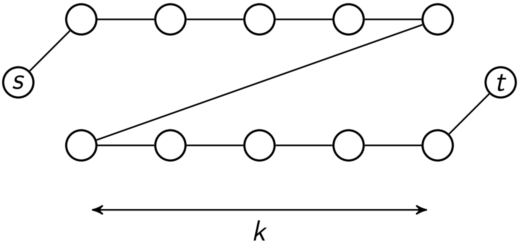

$$OPT \approx 3k$$

Integrality Gap

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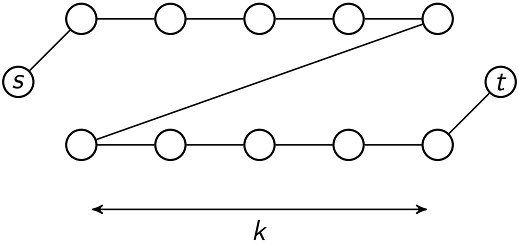


$$\frac{OPT}{OPT_{LP}} \to \frac{3}{2} \text{ as } k \to \infty$$

Graph Instances

Sebő and Vygen (2014) show that for graph TSP instances of s-t path TSP, can get a $\frac{3}{2}$-approximation algorithm (i.e. the algorithm produces a solution of cost at most $\frac{3}{2}OPT_{LP}$), so the integrality gap is tight for these instances.

We'll present a simplified version of this result due to Gao (2013).

Graph Instances

Given the input graph $G = (V, E)$ and an optimal solution, can replace any edge (i, j) in the optimal solution with the i-j path in G since these have the same cost.

So finding an optimal solution is equivalent to finding a multiset F of edges such that (V, F) is connected, $deg_F(s)$ and $deg_F(t)$ are odd, $deg_F(v)$ is even for all $v \in V - \{s, t\}$, and $|F|$ is minimum.

LP Relaxation

$$\text{Min} \quad \sum_{e \in E} x_e$$

subject to:

$$x(\delta(S)) \geq \begin{cases} 1, & |S \cap \{s,t\}| = 1, \\ 2, & |S \cap \{s,t\}| \neq 1, \end{cases}$$

$$x_e \geq 0, \qquad \forall e \in E.$$

Let x^* be an optimal LP solution.

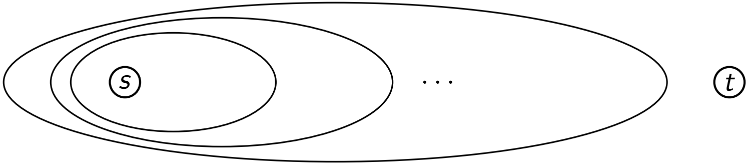
Narrow Cuts

As before, focus on *narrow cuts* S such that $x^*(\delta(S)) < 2$ (i.e. a τ-narrow cut for $\tau = 1$). Recall:

Theorem (An, Kleinberg, Shmoys (2012))

If S_1, S_2 are narrow cuts, $S_1 \neq S_2$, then either $S_1 \subset S_2$ or $S_2 \subset S_1$.

So the narrow cuts look like $s \in S_1 \subset S_2 \subset \cdots \subset S_k \subset V$.

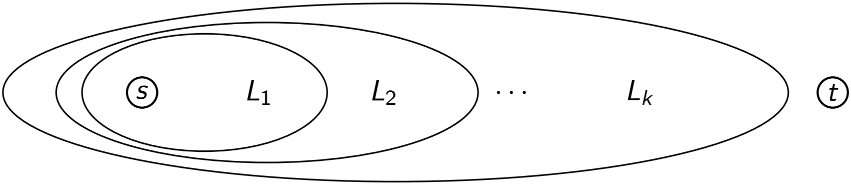


Let $S_0 \equiv \emptyset$, $S_{k+1} \equiv V$, $L_i \equiv S_i - S_{i-1}$.

Key Idea

Find a tree spanning L_i in the support of x^* for each i. Connect each of these via a single edge from L_i to L_{i+1}. Let F be the resulting tree, T the vertices in F whose parity needs changing.

Then $|F| = n - 1$ and $|\delta(S_i) \cap F| = 1$ for each narrow cut S_i.



Key Lemma

Recall:

Lemma

Let S be an odd set. If $|S \cap \{s,t\}| = 1$, then $|F \cap \delta(S)|$ is even.

$$\begin{array}{lll} & \text{Min} \quad \sum_{e \in E} c_e x_e & \\ \text{subject to:} & x(\delta(S)) \geq 1, & \forall S \subseteq V, |S \cap T| \text{ odd} \\ & x_e \geq 0, & \forall e \in E. \end{array}$$

Lemma

$y = \frac{1}{2}x^*$ is feasible for the the T-join LP.

Gao (2013)

Theorem (Gao (2013))

For spanning tree F constructed by the algorithm, let J be a minimum-cost T-join. Then $c(F \cup J) \leq \frac{3}{2} OPT_{LP}$.

$$\text{Min} \quad \sum_{e \in E} x_e$$

subject to:

$$x(\delta(S)) \geq \begin{cases} 1, & |S \cap \{s,t\}| = 1, \\ 2, & |S \cap \{s,t\}| \neq 1, \end{cases}$$

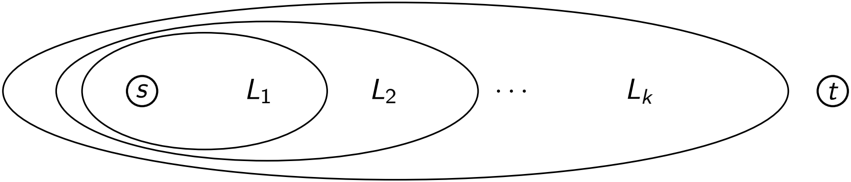
$$x_e \geq 0, \qquad \forall e \in E.$$

Last Lemma

Let $E(x^*) = \{e \in E : x^*_e > 0\}$ be the *support* of LP solution x^*, $H = (V, E(x^*))$ the support graph of x^*, $H(S)$ the graph induced by a set S of vertices.

Lemma (Gao (2013))

For $1 \leq p \leq q \leq k+1$, $H\left(\bigcup_{p \leq i \leq q} L_i\right)$ *is connected.*



The Big Question

Is there a $\frac{3}{2}$-approx. alg. for s-t path TSP for general costs?

One Idea

Idea: Construct a spanning tree F just as in Gao's algorithm for the graph case. Then again $y = \frac{1}{2}x^*$ is feasible for the T-join LP, and the overall cost of F plus the T-join is at most $c(F) + \frac{1}{2}\sum_{e \in E} c_e x_e^*$.

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Idea: Construct a spanning tree F just as in Gao's algorithm for the graph case. Then again $y = \frac{1}{2}x^*$ is feasible for the T-join LP, and the overall cost of F plus the T-join is at most $c(F) + \frac{1}{2}\sum_{e\in E} c_e x_e^*$.

Problem: Not clear how to bound the cost of F. Gao (2014) has an example showing that F can have cost greater than OPT_{LP}.

The Bigger Question

Best-of-Many Christofides' is *provably* better than Christofides' for s-t path TSP. What about the standard TSP?

An empirical answer

Did some computational work with Cornell CS undergraduate Kyle Genova to see whether Best-of-Many Christofides is any better than standard Christofides in practice. Paper to appear in upcoming ESA.

The algorithms

We implement algorithms to do the following:

- Run the standard Christofides' algorithm (Christofides 1976);
- Construct explicit convex combination via column generation (An 2012);
- Construct explicit convex combination via *splitting off* (Frank 2011, Nagamochi, Ibaraki 1997);
- Add sampling scheme *SwapRound* to both of above; gives negative correlation properties (Chekuri, Vondrák, Zenklusen 2010);
- Compute and sample from *maximum entropy distribution* (Asadpour, Goemans, Madry, Oveis Gharan, Saberi 2010).

The experiments

The algorithms were implemented in C++, run on a machine with a 4.00Ghz Intel i7-875-K processor with 8GB DDR3 memory.

We run these algorithms on several types of instances:

- 59 Euclidean TSPLIB (Reinelt 1991) instances up to 2103 vertices (avg. 524);
- 5 non-Euclidean TSPLIB instances (gr120, si175, si535, pa561, si1032);
- 39 Euclidean VLSI instances (Rohe) up to 3694 vertices (avg. 1473);
- 9 graph TSP instances (Kunegis 2013) up to 1615 vertices (avg. 363).

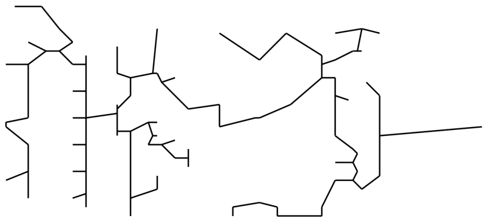
The results

	Std	ColGen		ColGen+SR	
		Best	Ave	Best	Ave
TSPLIB (E)	9.56%	4.03%	6.44%	3.45%	6.24%
VLSI	9.73%	7.00%	8.51%	6.40%	8.33%
TSPLIB (N)	5.40%	2.73%	4.41%	2.22%	4.08%
Graph	12.43%	0.57%	1.37%	0.39%	1.29%

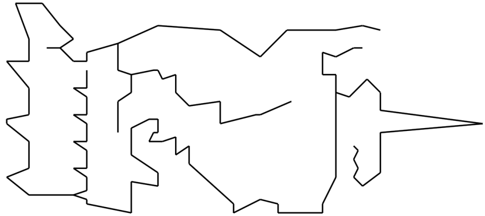
	MaxEnt		Split		Split+SR	
	Best	Ave	Best	Ave	Best	Ave
TSPLIB (E)	3.19%	6.12%	5.23%	6.27%	3.60%	6.02%
VLSI	5.47%	7.61%	6.60%	7.64%	5.48%	7.52%
TSPLIB (N)	2.12%	3.99%	2.92%	3.77%	1.99%	3.82%
Graph	0.31%	1.23%	0.88%	1.77%	0.33%	1.20%

Costs given as percentages in excess of optimal.

The results



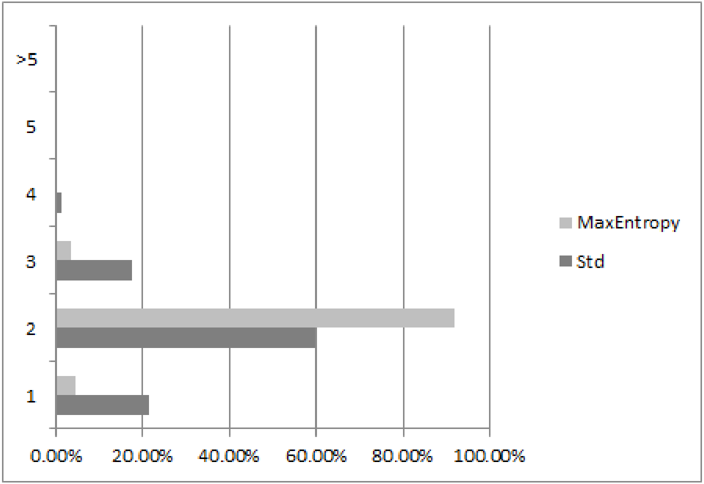
Standard Christofides MST (Rohe VLSI instance XQF131)



Splitting off + SwapRound

The results

BoMC yields more vertices in the tree of degree two.



The results

So while the tree costs more (as percentage of optimal tour)...

	Std	BOM
TSPLIB (E)	87.47%	98.57%
VLSI	89.85%	98.84%
TSPLIB (N)	92.97%	99.36%
Graph	79.10%	98.23%

The results

...the matching costs much less.

	Std	CG	CG+SR	MaxE	Split	Sp+SR
TSPLIB (E)	31.25%	11.43%	11.03%	10.75%	10.65%	10.41%
VLSI	29.98%	14.30%	14.11%	12.76%	12.78%	12.70%
TSPLIB (N)	24.15%	9.67%	9.36%	8.75%	8.77%	8.56%
Graph	39.31%	5.20%	4.84%	4.66%	4.34%	4.49%

Conclusion

Q: Are there empirical reasons to think BoMC might be provably better than Christofides' algorithm?

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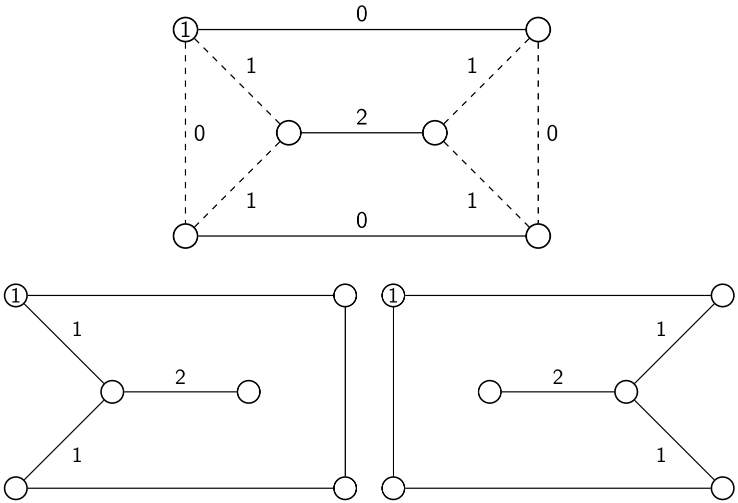
A: Yes.

Maximum entropy sampling, or splitting off with SwapRound seem like the best candidates.

Conclusion

However, we have to be careful, as the following, very recent, example of Schalekamp and van Zuylen shows.



Conclusions

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Vygen (2015) also uses careful construction to improve s-t path TSP from 1.6 to 1.5999.

If we want to use the best sample from Max Entropy or SwapRound, then might need to prove some tail bounds.