Lecture 2: Asymmetric TSP

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Two Basic Versions

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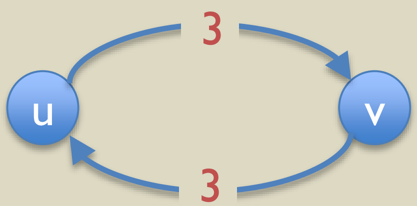
SYMMETRIC: distance from u to v equals distance from v to u



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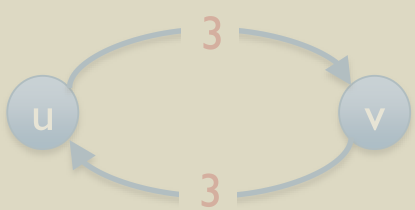
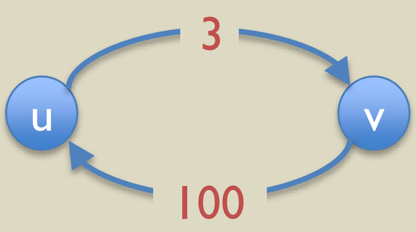


ASYMMETRIC: more general and no such assumption is made



Motivation of *Asymmetric* TSP

Approximation algorithms for ATSP

What is the best possible algorithm?

1980's

Frieze, Galbiati & Maffiolo:

$log_2(n)$**-approximation** for **Asymmetric** TSP

1990's

• • •

2000's

Bläser'03:

$0.99 log_2(n)$-approximation for ATSP

Kaplan, Lewenstein, Shafrir & Sviridenko'05:

$0.84 log_2(n)$-approximation for ATSP

Feige & Singh'07:

$0.67 log_2(n)$-approximation for ATSP

2010's

Asadpour, Goemans, Madry, Oveis Gharan, and Saberi:

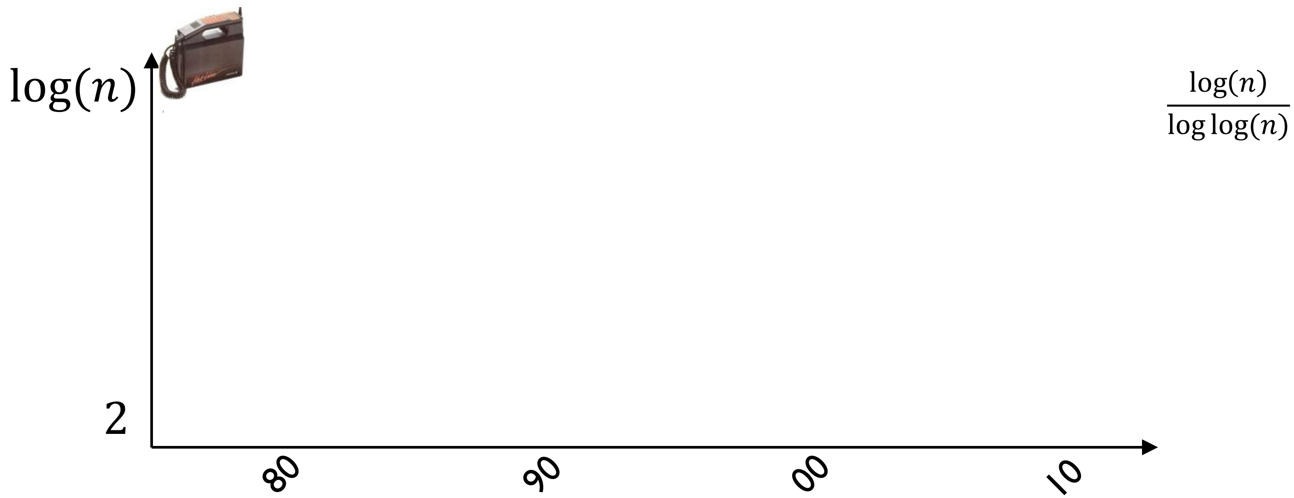
- **New approach** relating ATSP to the **thin tree** graph problem
- $\boldsymbol{O\left(\frac{log(n)}{loglog(n)}\right)}$**-approximation** for ATSP

Today

Major open problem to understand the approximability of ATSP

- **NP-hard** to approximate ATSP within 75/74
- Best algorithm has super constant approximation guarantee
- Held-Karp relaxation conjectured to give **2-approximation**

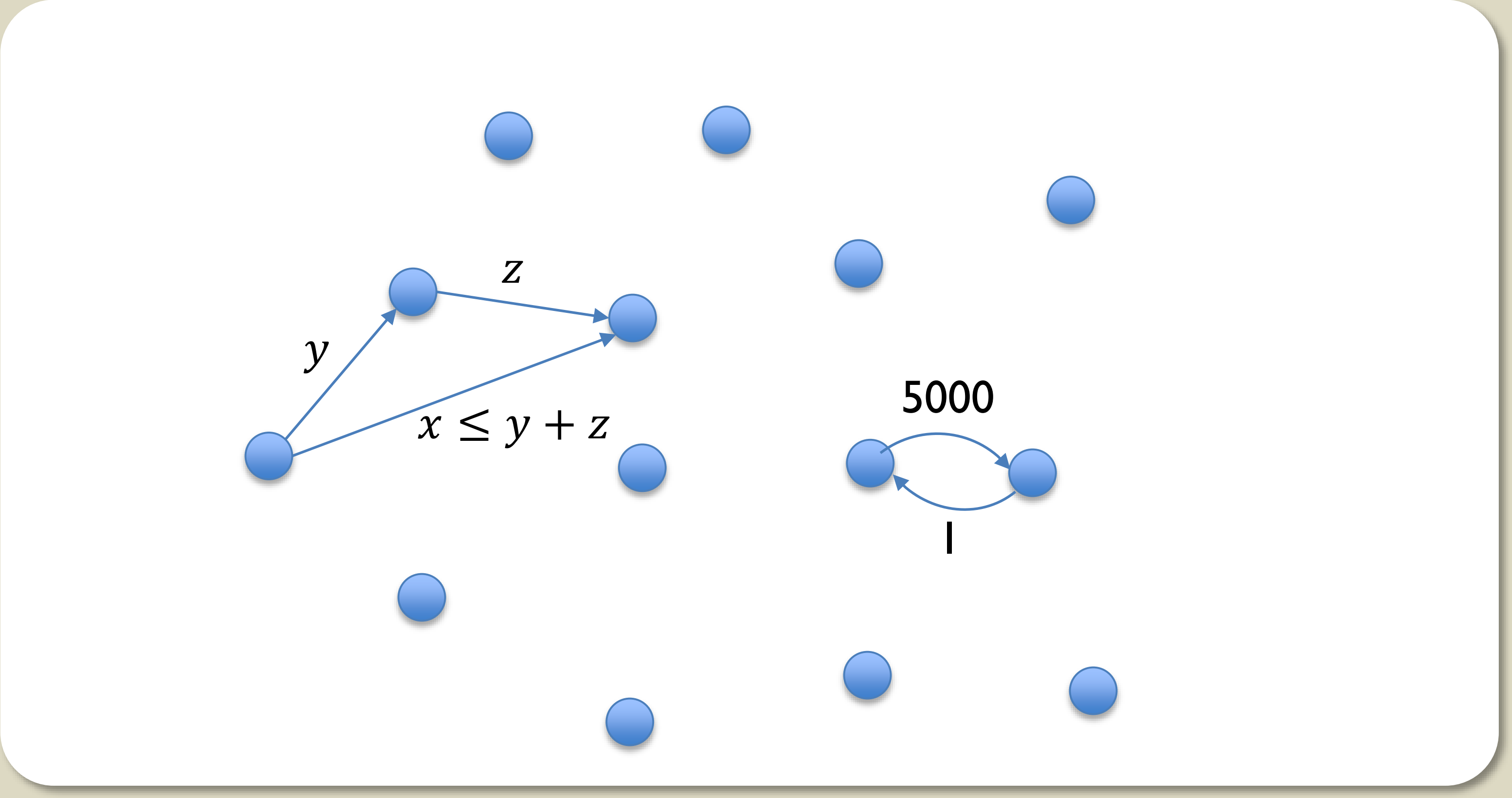
Progress on ATSP vs mobiles



Asymmetric Traveling Salesman Problem

INPUT: a complete digraph $G = (V, E)$ with pairwise (*not necessarily symmetric*) distances that satisfy the triangle inequality

OUTPUT: a tour of minimum weight that visits each vertex once



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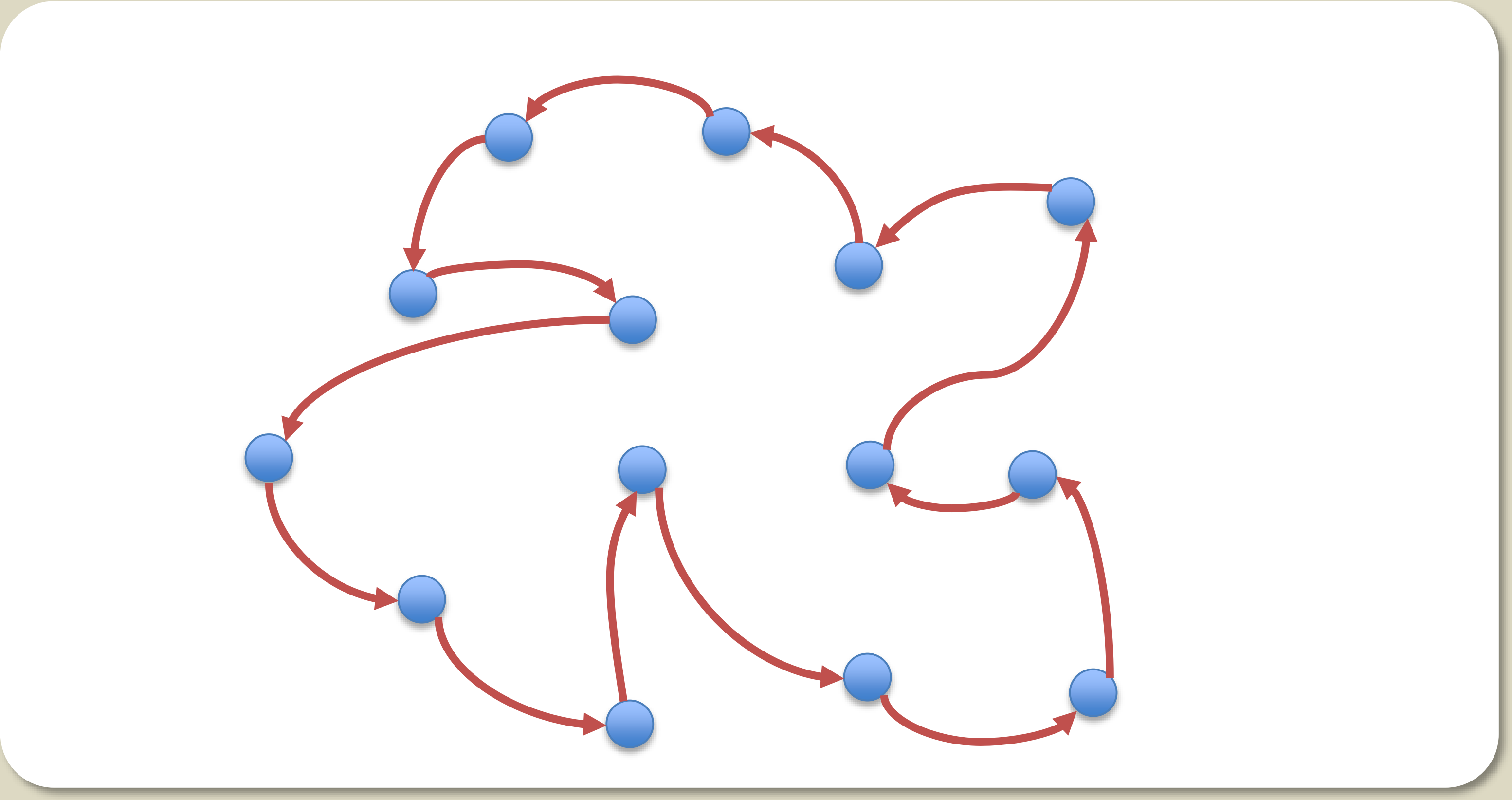


WHAT'S THE DEAL:

CAN'T WE JUST GENERALIZE CHRISTOFIDES???

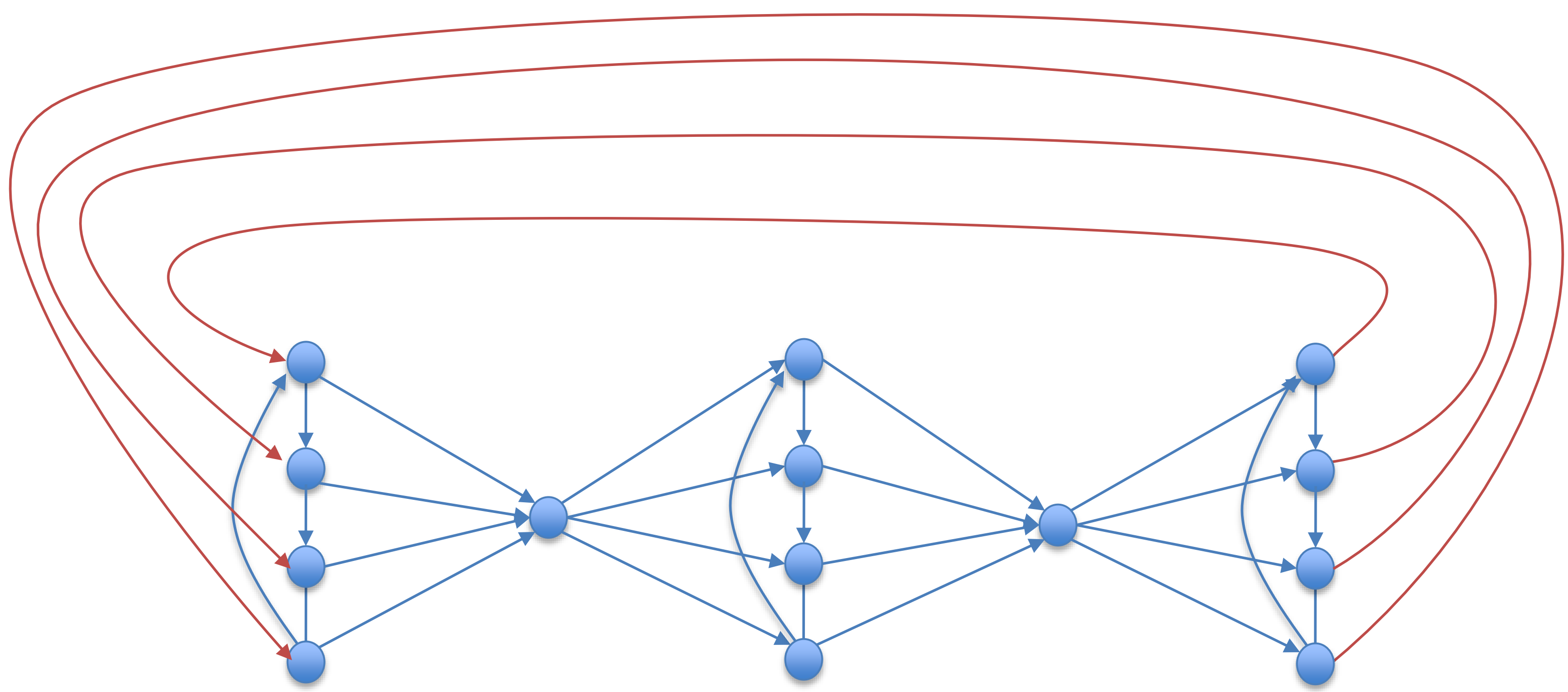
Spanning Tree Approach

Find (undirected) spanning tree

Obtain Eulerian graph via a circulation that sends one unit through each tree edge

Blue edges have cost 1, red edges have cost $M \gg 1$

So we wish to find tour with minimum number of red edges



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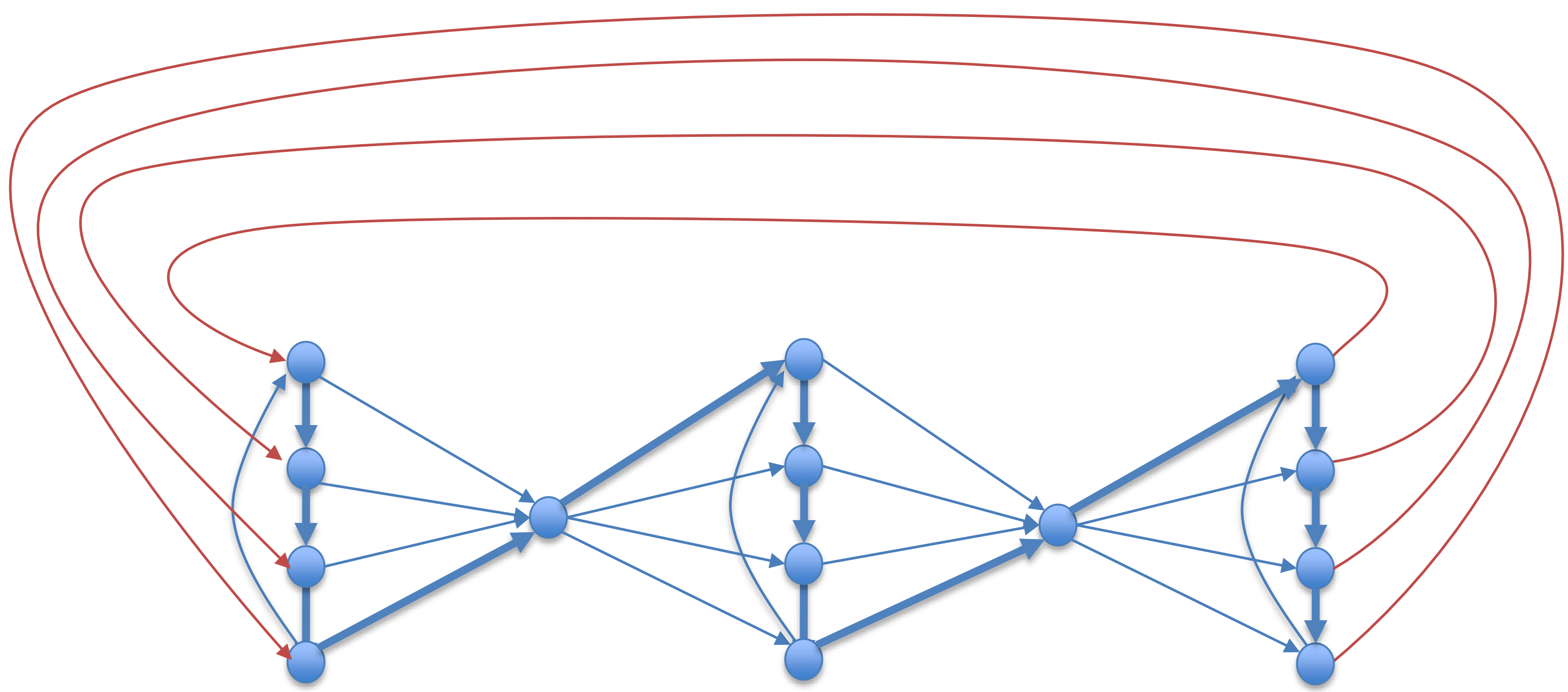

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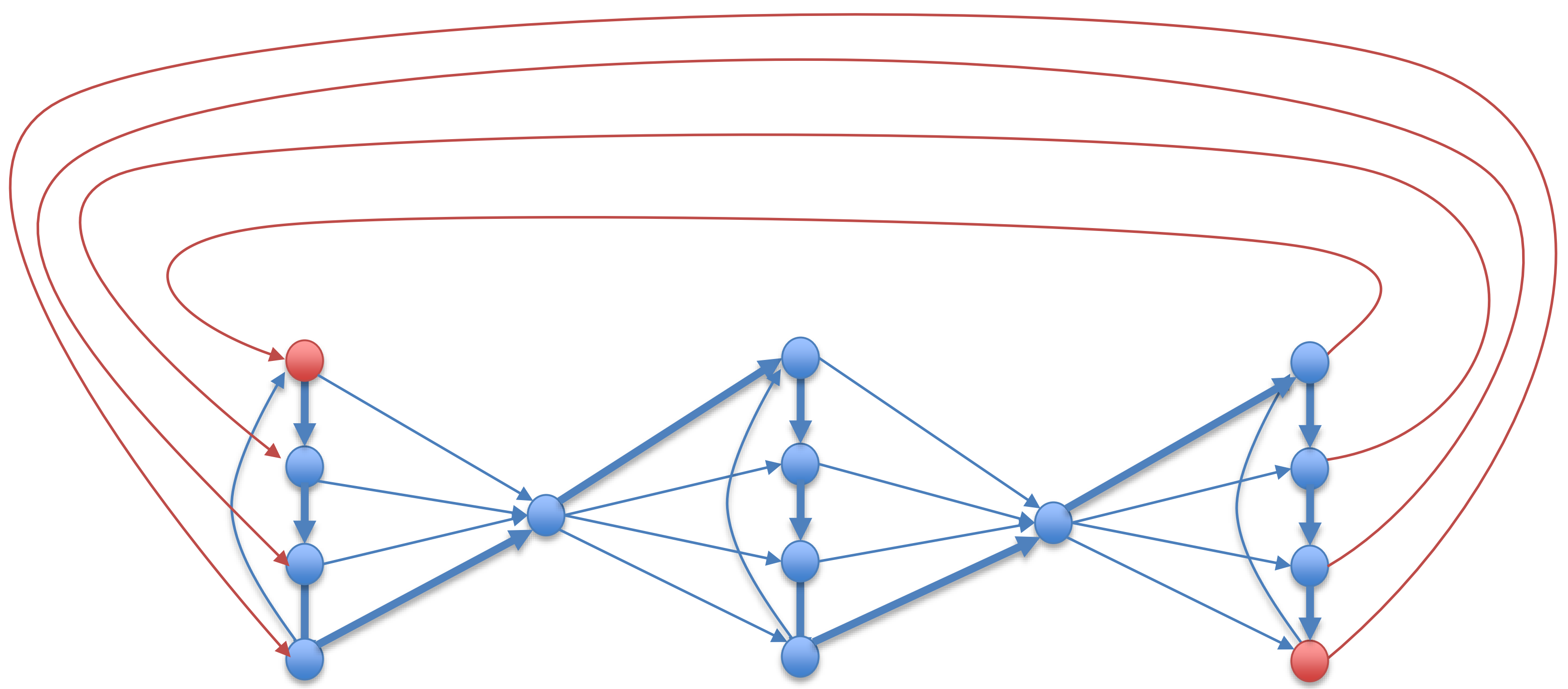

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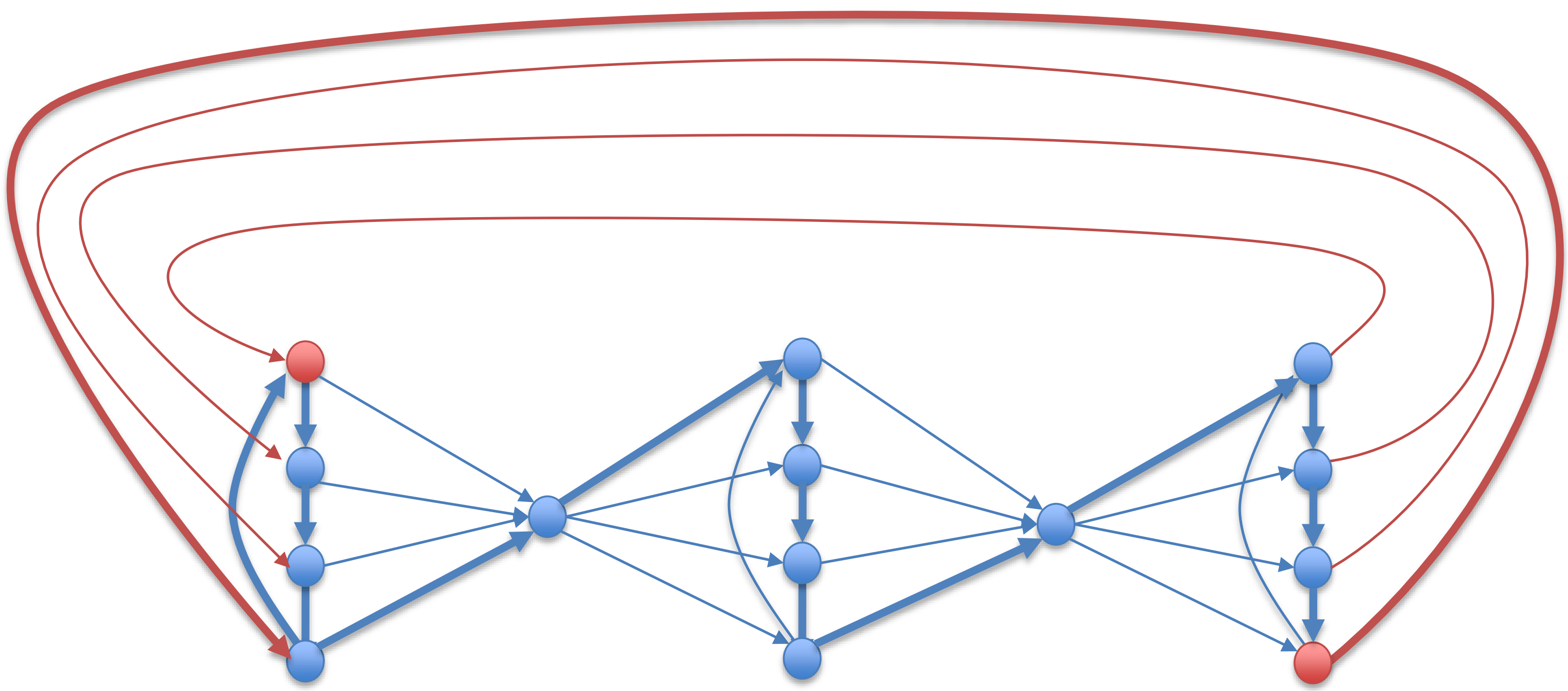

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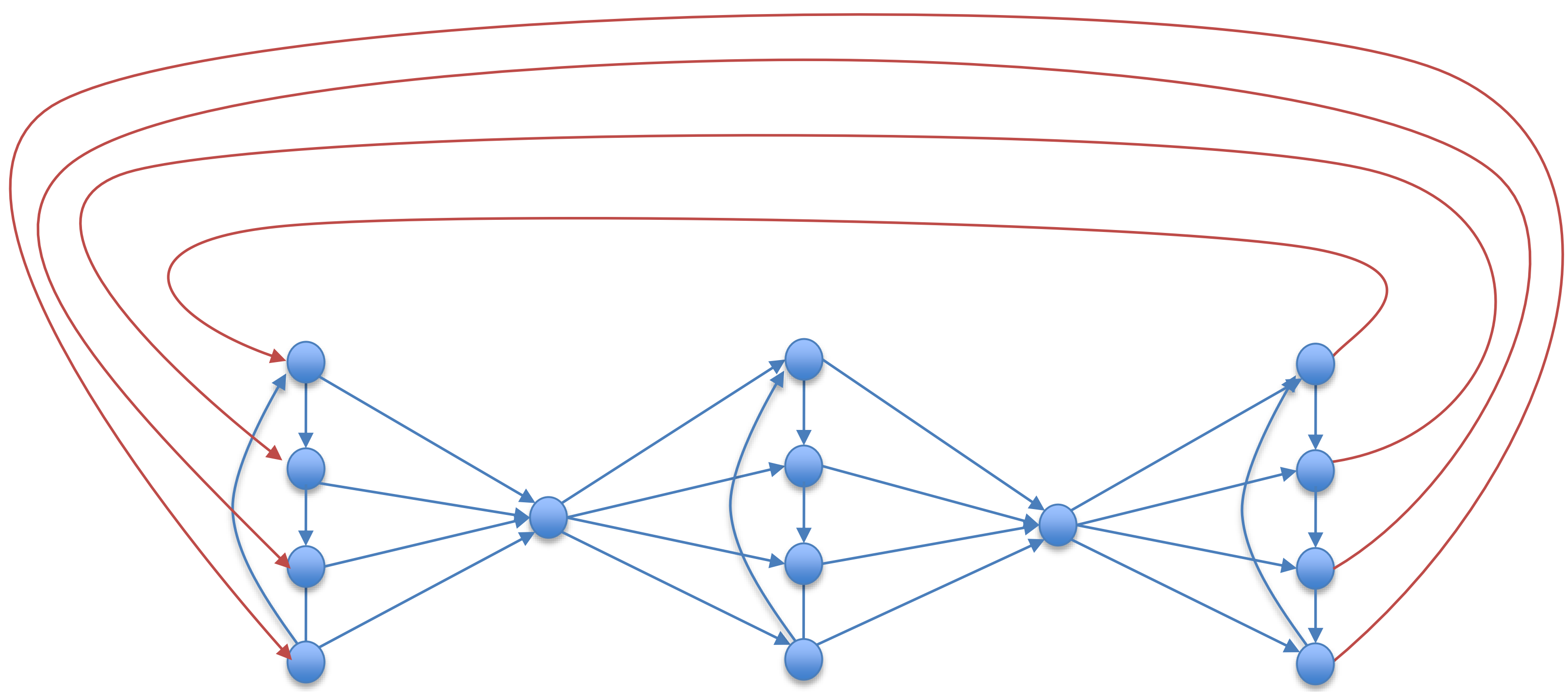

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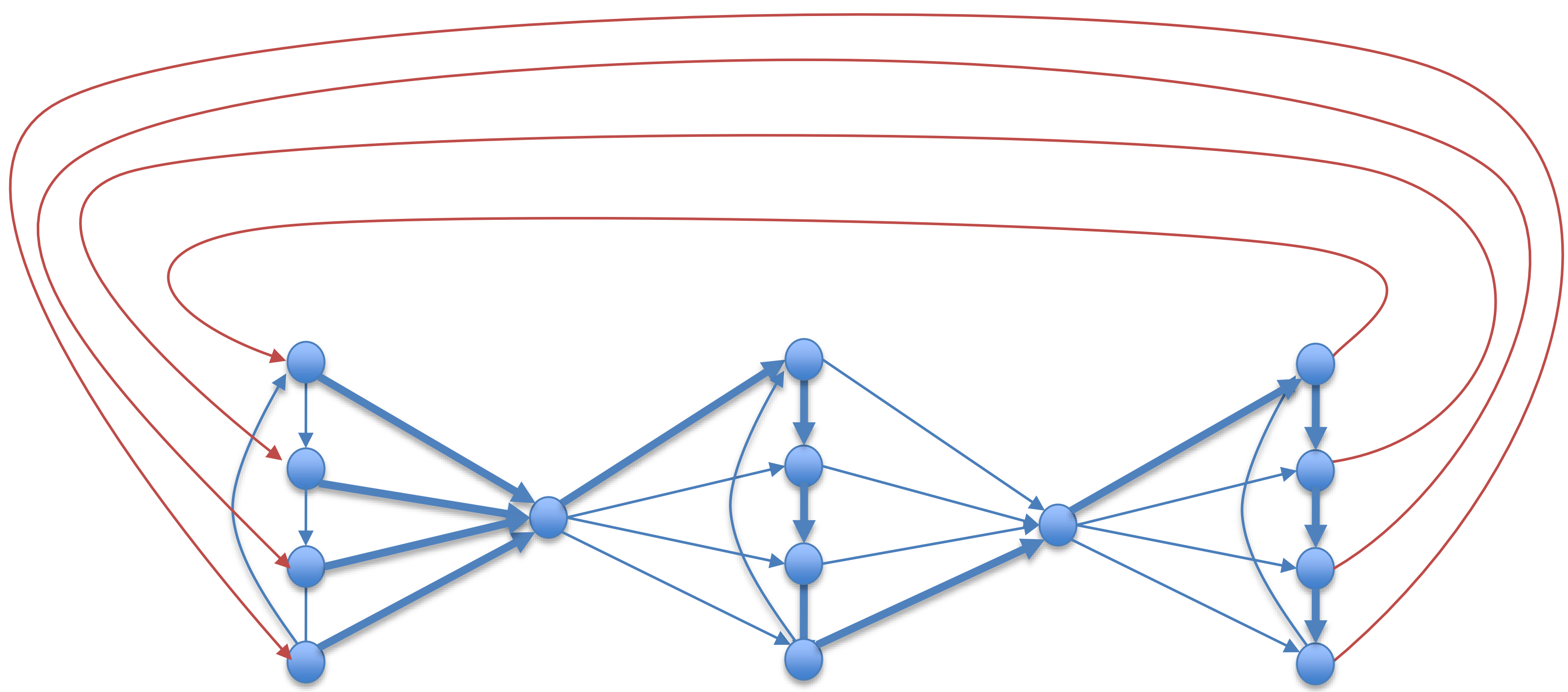

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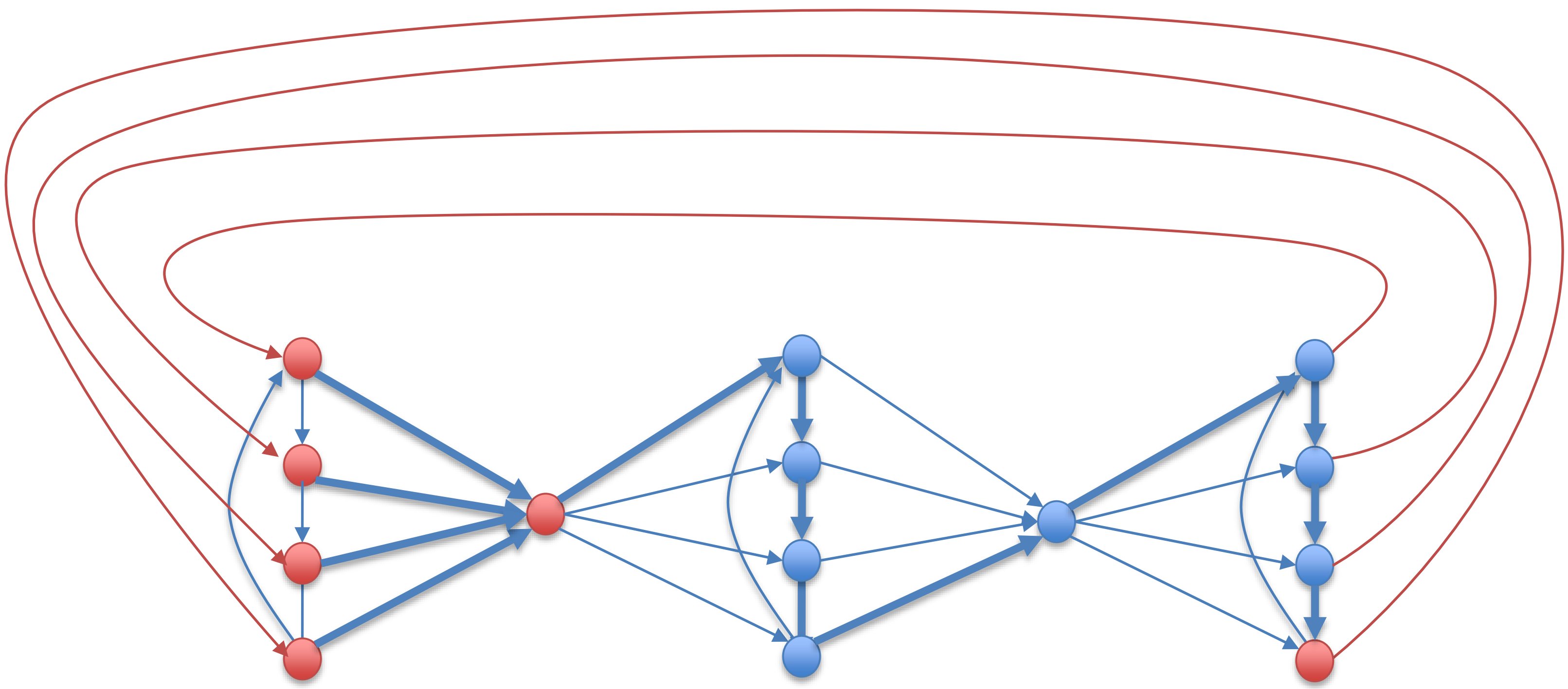

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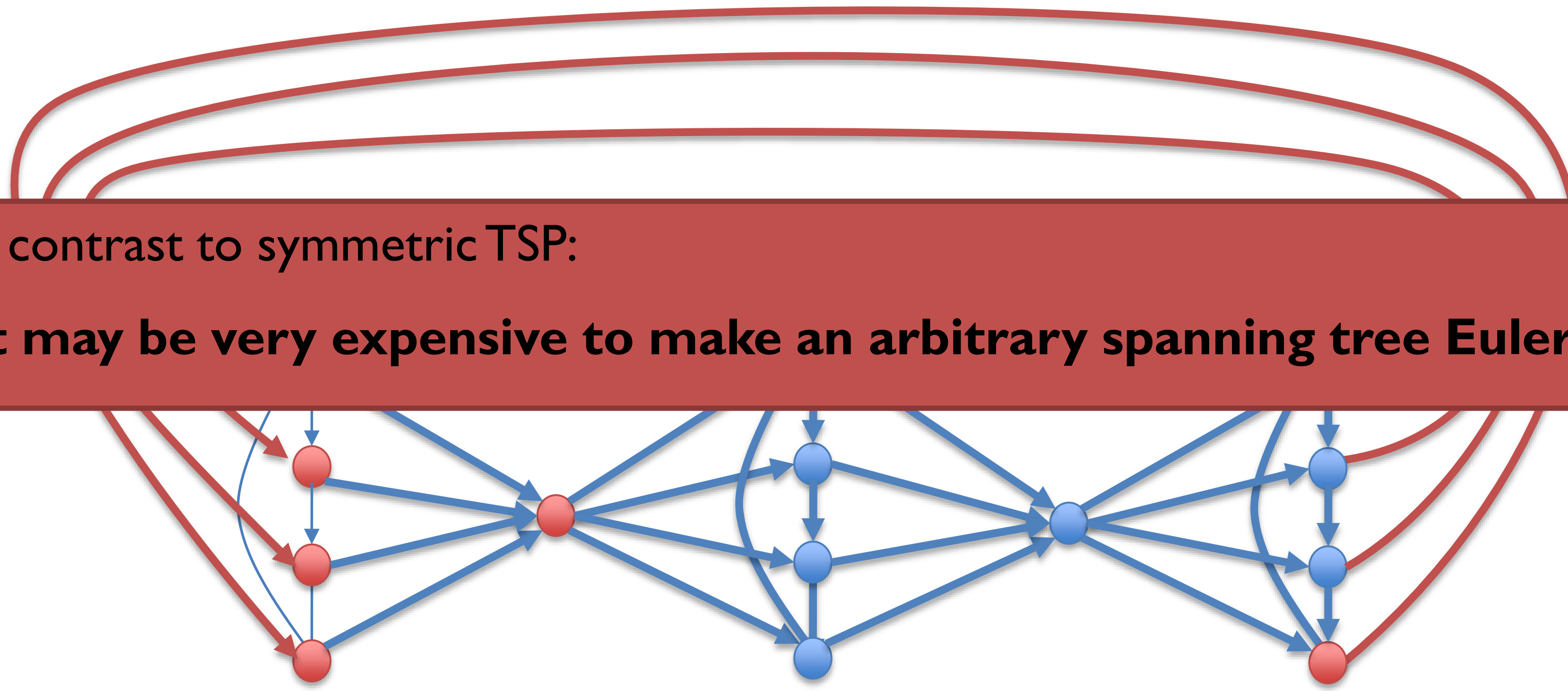

In contrast to symmetric TSP:

It may be very expensive to make an arbitrary spanning tree Eulerian

Cycle Cover

FIRST APPROXIMATION ALGORITHM FOR ATSP

What is a lower bound on OPT?

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The weight of a minimum weight cycle-cover is at most OPT

A cycle cover is a collection of cycles so that each vertex is in exactly one cycle

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PROOF:

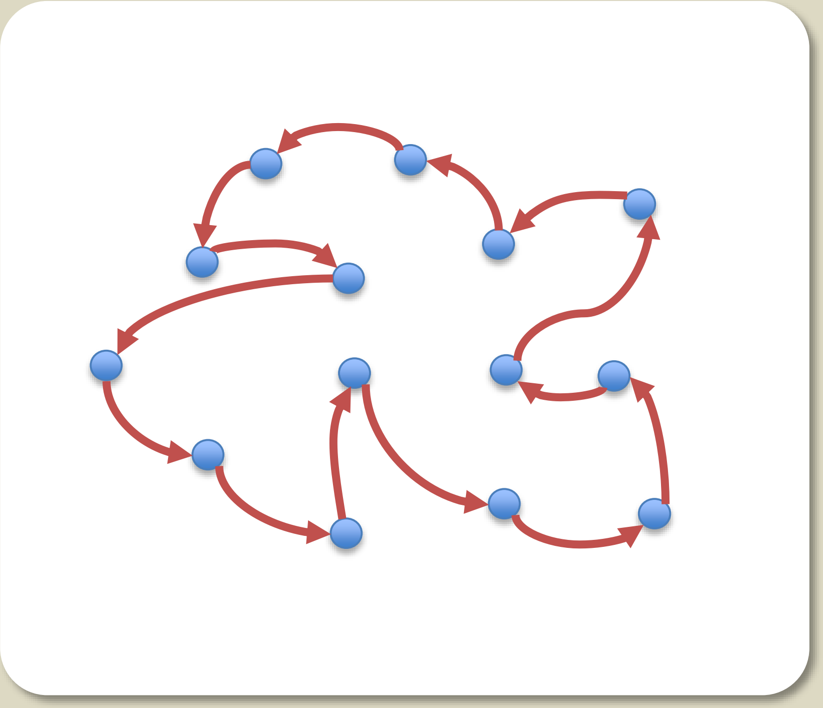
What is a lower bound on OPT?

The weight of a minimum weight cycle-cover is at most OPT

A cycle cover is a collection of cycles so that each vertex is in exactly one cycle

PROOF:

- Optimal tour is a cycle cover
- Hence, minimum cycle cover has weight at most OPT



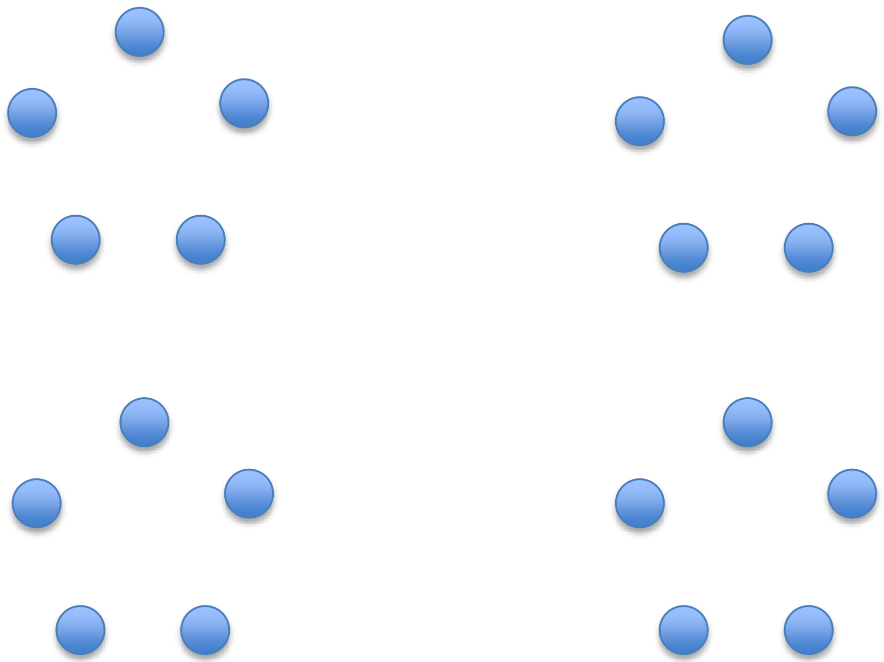
A minimum cycle cover can be computed in polynomial time

Cycle Cover Algorithm

Find min-cost cycle cover

Select a representative in each component

Repeat on representatives until graph is connected



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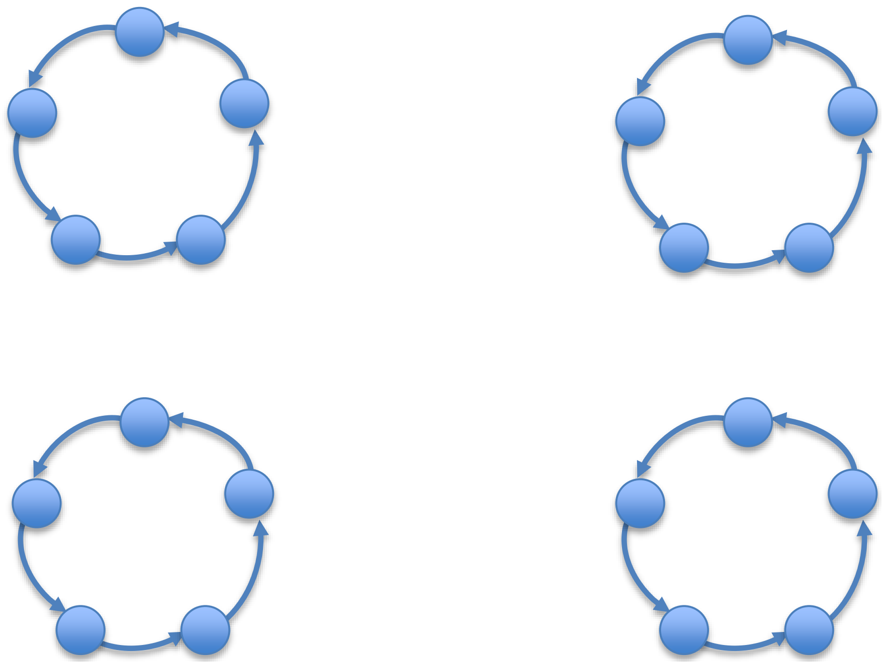

Cost of cycle cover $\leq OPT$

Cycle Cover Algorithm

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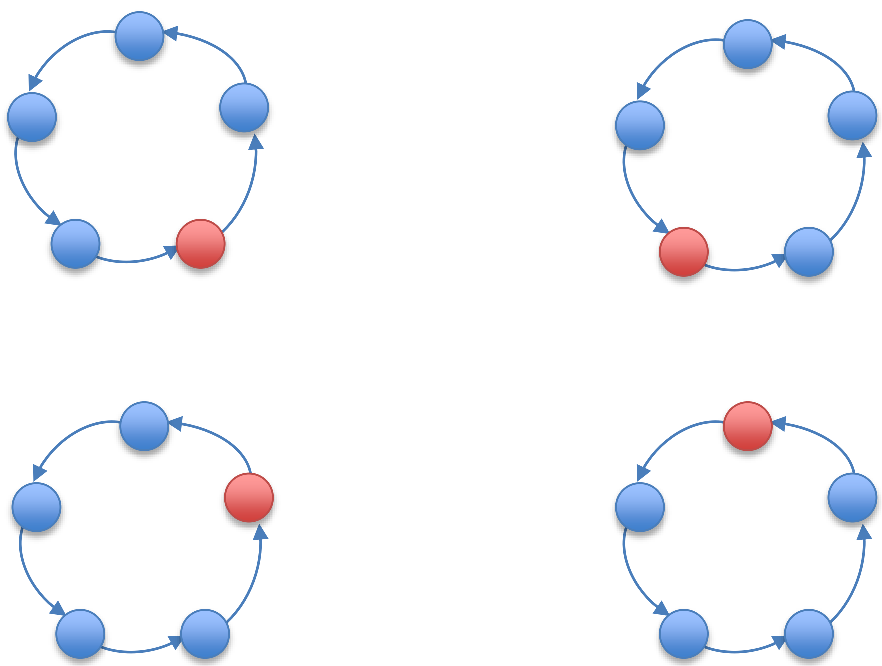

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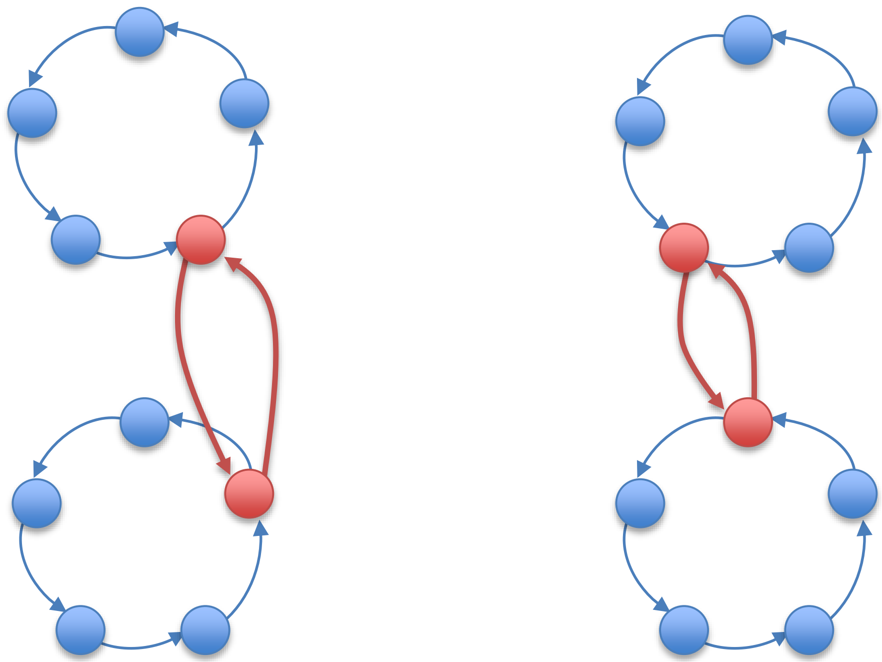

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Why?

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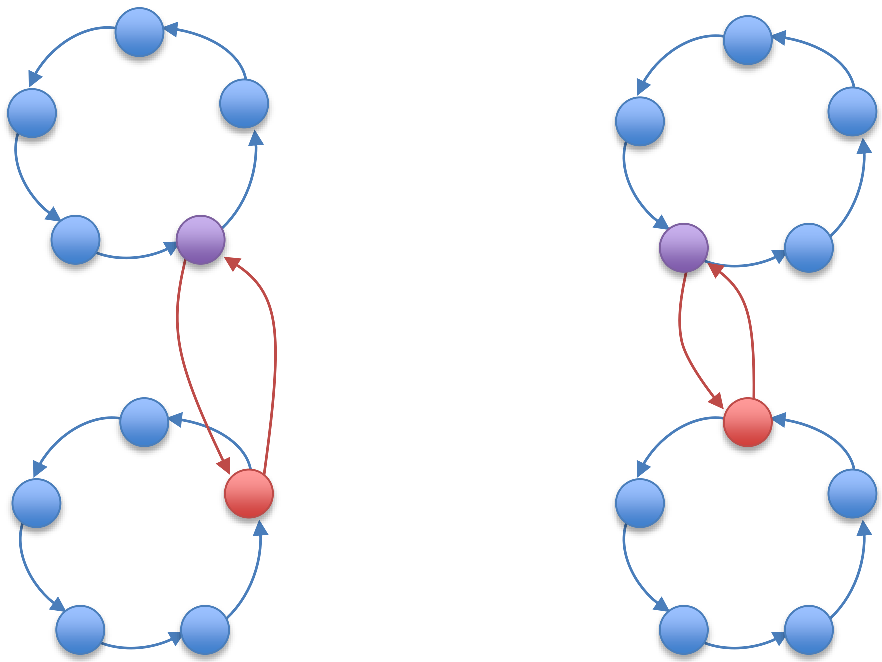

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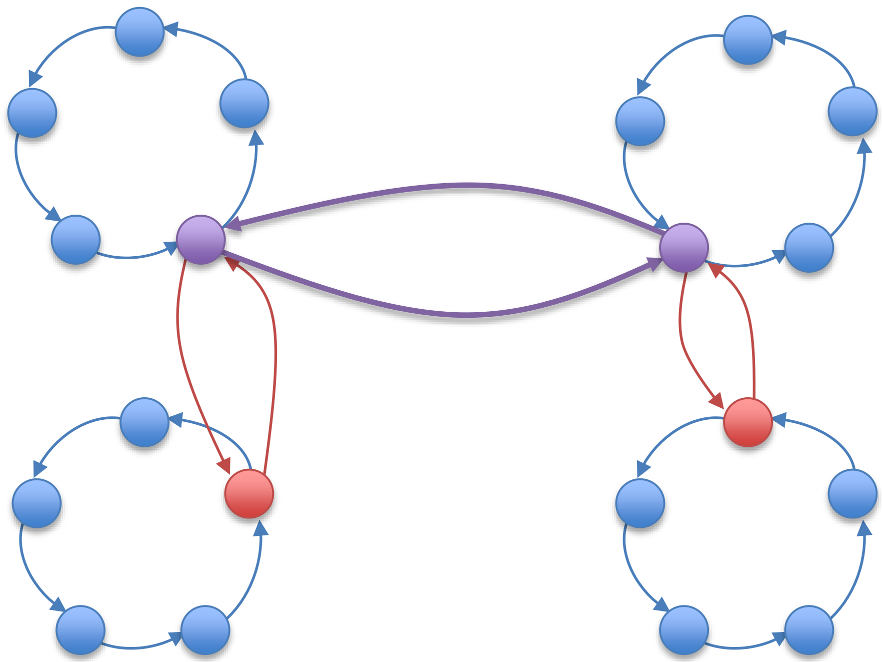

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Cycle Cover Algorithm

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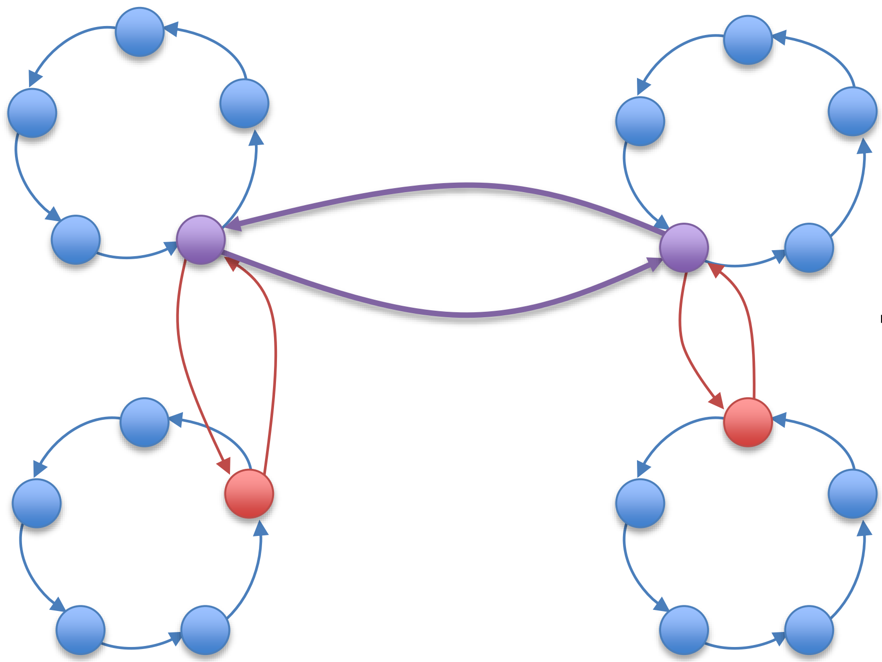
Worst case: all cycles have length 2 so we need to repeat $\log_2 n$ times (each time cost OPT_{LP})



Cost of cycle cover $\leq OPT$

Cost of cycle cover $\leq OPT$

Cost of cycle cover $\leq OPT$

Total cost $\leq 3 \cdot OPT$

Cycle Cover Technique

THEOREM: Frieze, Galbiati, Maffiolo'82

ATSP has a $\log_2 n$ -approximation algorithm

- $0.99 \log_2 n$-approximation algorithm [Bläser'03]
- $0.84 \log_2 n$-approximation algorithm [Kaplan, Lewenstein, Shafrir, Sviridenko'05]
- $0.67 \log_2 n$-approximation algorithm [Feige, Singh'07]

BETTER APPROXIMATION ALGORITHMS ARE BASED ON THE HELD-KARP RELAXATION

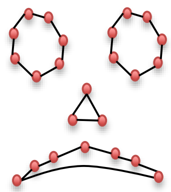
Held-Karp Relaxation of ATSP

Variables: x_{uv} = "indicate whether arc (u, v) is used in tour"

Held-Karp Relaxation

Minimize: $\sum_{uv \in E} w(u,v)\, x_{uv}$

Subject to: $x(\delta^+(v)) = x(\delta^-(v)) = 1$ for all $v \in V$



$x(\delta^+(S)) \geq 1$ for all $S \subset V$

$x \geq 0$

Easy to Find Eulerian graph

Easy to Find Connected Graph

TOOLS FOR ROUNDING LP

Circulations

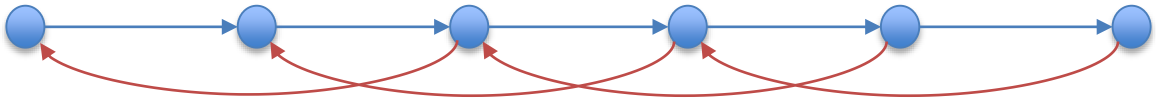
INPUT: a digraph $G = (V, E)$ and for each arc $e \in E$

a lower bound $l(e) \geq 0$ and an upper bound $u(e) \geq 0$

OUTPUT: a circulation $f: E \rightarrow R_+$ satisfying

flow conservation: $f\left(\delta^+(v)\right) = f\left(\delta^-(v)\right)$ for each $v \in V$

edge bounds: $l(e) \leq f(e) \leq u(e)$ for each $e \in E$



Blue edges have lower bound 1 and upper bound 2

Red edges have lower bound 0 and upper bound ∞

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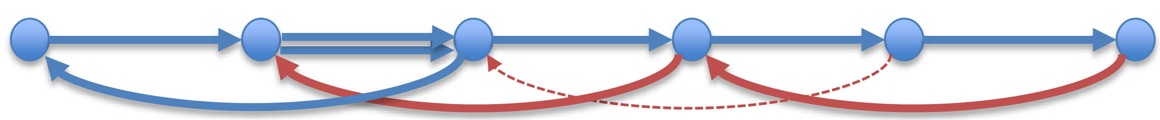


Blue edges have lower bound 1 and upper bound 2

Red edges have lower bound 0 and upper bound ∞

Can be calculated in polytime and also a min cost circulation can be found

When does a circulation exist?

NECESSARY CONDITIONS:

- For every arc $e \in E$ we must have $l(e) \leq u(e)$
- For every $S \subset V$ we must have $l\big(\delta^-(S)\big) \leq u\big(\delta^+(S)\big)$

HOFFMAN's CIRCULATION THEOREM: Hoffman'60

The above conditions are also **sufficient.**

Furthermore, if l and u are integer valued, the circulation f can be chosen to be integral.

Basic idea select a subset of edges and make it Eulerian by finding a circulation

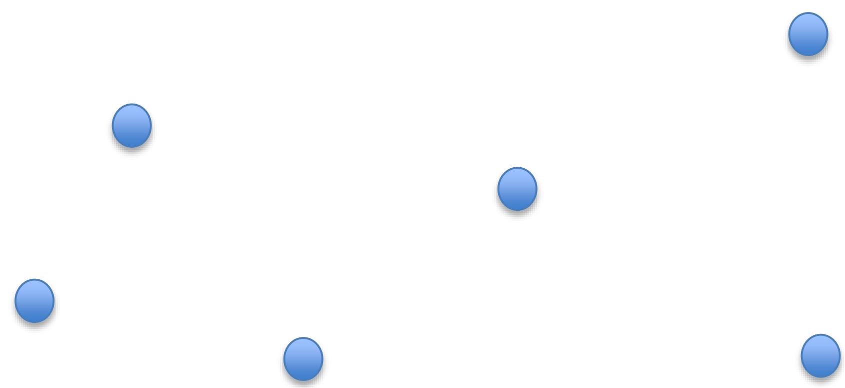
RANDOMIZED LP ROUNDING

Randomized Round

Find an optimal solution x^* to LP relaxation

Form H by taking each edge with probability equal to its LP-value

Compute Eulerian graph by finding an (integral) min cost circulation with lower bound 1 for each arc in H



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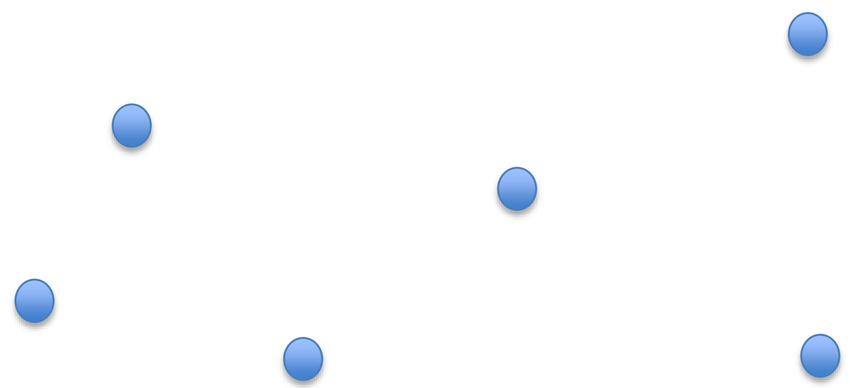

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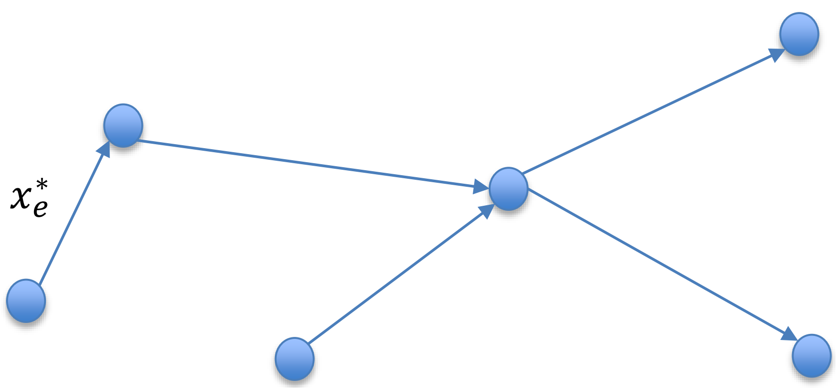
Taken with probability x_e^*



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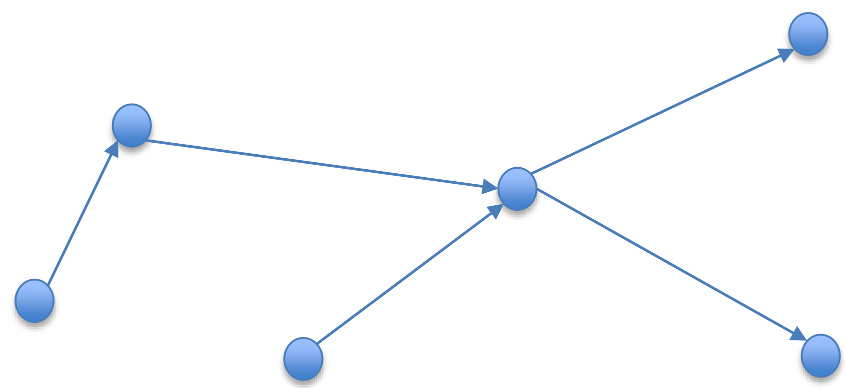

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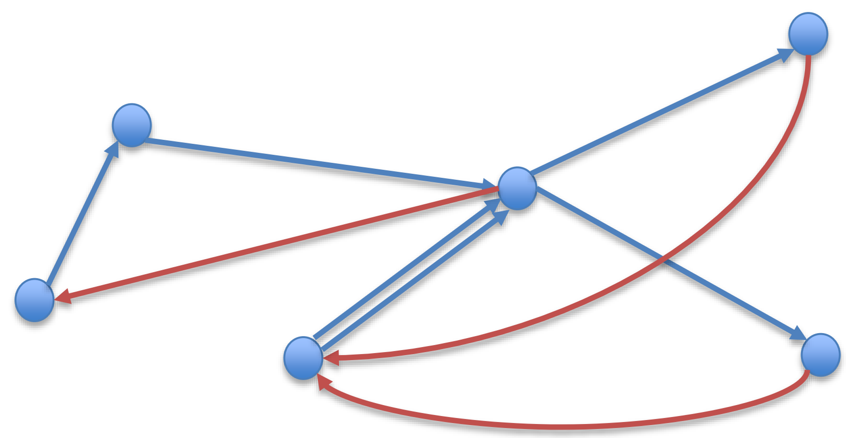

What's the problem?

- With high probability the sampled graph H is not even connected
- So we will return an Eulerian graph but it is **not** connected ☹

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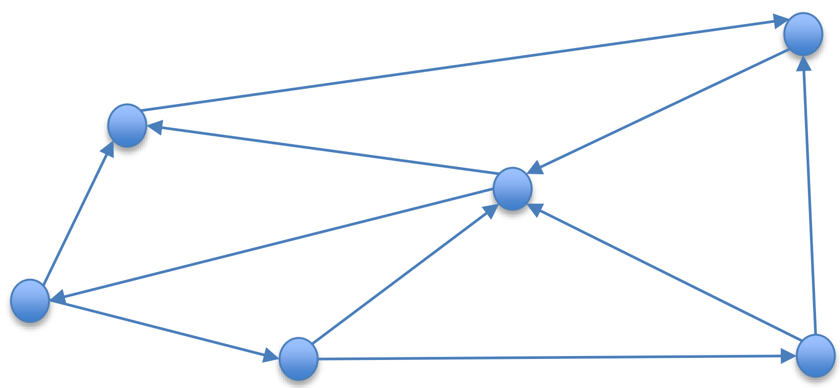


Well connected?

Eulerian?

Analyzing H: out-degree of a vertex

What is the expected out-degree of v?

It is $K \cdot x^*\big(\delta^+(v)\big) = 1000 \ln n =: \mu$

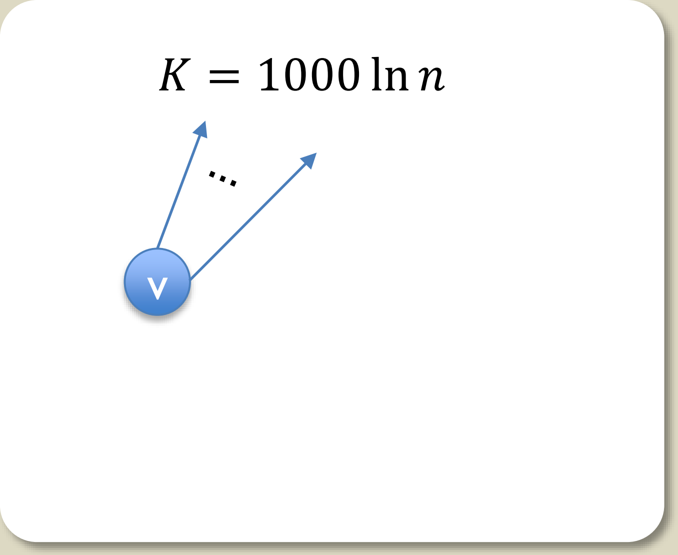
The number of outgoing edges is the sum of random independent 0/1 variables

Hence, by standard Chernoff bound

$$\Pr\left[\left|\left|\delta_H^+(v)\right| - \mu\right| \geq \frac{\mu}{3}\right] \leq e^{-\frac{\mu}{30}} \leq \frac{1}{2n^{10x^*(\delta^+(v))}}$$

In words: the number of edges will deviate

from its expectation more than a fraction 1/3 with probability at most $O(\frac{1}{n^{10}})$



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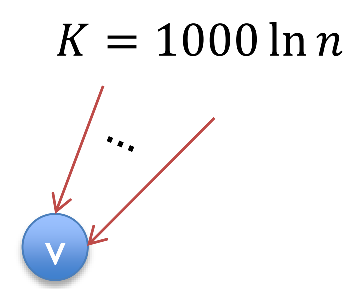

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Analyzing H: bad cuts

We say that a cut S is bad in H if the incoming and outgoing edges deviate more than a fraction 1/3

S is bad in H if

$$\left||\delta_H^-(S)| - \mu\right| \geq \frac{\mu}{3} \quad \text{or} \quad \left||\delta_H^+(S)| - \mu\right| \geq \frac{\mu}{3}$$

where $\mu = x^*\left(\delta^-(S)\right) = x^*(\delta^+(S))$

By previous calculations

$$\Pr[S\ is\ bad] \leq \Pr\left[\left||\delta_H^-(v)| - \mu\right| \geq \frac{\mu}{3}\right] + \Pr\left[\left||\delta_H^+(v)| - \mu\right| \geq \frac{\mu}{3}\right]$$

$$\leq \frac{1}{n^{10x^*(\delta^+(S))}}$$

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By previous calculations

$$\mathbf{Pr}[\boldsymbol{S\ is\ bad}]$$

$$\leq \frac{\mathbf{1}}{\boldsymbol{n}^{\mathbf{10}\boldsymbol{x}^*(\boldsymbol{\delta}^+(\boldsymbol{S}))}}$$

Probability that H is good?

$\boldsymbol{H}$ is good if no cut is bad

We know that a single cut S is bad w.p $\leq \frac{1}{n^{10x^*\left(\delta^+(S)\right)}} \leq \frac{1}{n^{10}}$

But 2^n many cuts so we can't make union bound…

Remedy: beautiful result by Karger

BOUNDING NUMBER OF SMALL CUTS: Karger

Consider an undirected graph $G = (V, E)$ with edge-weights w. Let c be the value of a min-cut.

Then the number of cuts of value αc is at most $\leq n^{2\alpha}$

But our graph is directed, **why can we still use the above theorem?**

It is Eulerian (w.r.t. to weights x^*) so any cut of out-degree/in-degree c corresponds to a cut of value 2c in the undirected graph

Probability that H is good?

- Cuts of value [1,2] at most $n^{2\cdot 2}$ many Prob. that such a cut is bad $n^{-10\cdot 1}$
- Cuts of value [2,3] at most $n^{2\cdot 3}$ many Prob. that such a cut is bad $n^{-10\cdot 2}$
- Cuts of value [3,4] at most $n^{2\cdot 4}$ many Prob. that such a cut is bad $n^{-10\cdot 3}$
- Cuts of value [4,5] at most $n^{2\cdot 5}$ many Prob. that such a cut is bad $n^{-10\cdot 4}$

- Cuts of value [n-1, n] at most $n^{2\cdot n}$ many Prob. that such a cut is bad $n^{-10\cdot(n-1)}$

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By union bound,

$$\Pr[H\ is\ good] \geq 1 - \sum_{i=1}^{n-1} n^{2\cdot(i+1)-10i}$$

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By union bound,

$$\Pr[H\ is\ good] \geq 1 - \sum_{i=1}^{n-1} n^{2\cdot(i+1)-10i} = 1 - \sum_{i=1}^{n-1} \frac{n^2}{n^{8i}}$$

Probability that H is good?

- Cuts of value [1,2] at most $n^{2\cdot 2}$ many — Prob. that such a cut is bad $n^{-10\cdot 1}$
- Cuts of value [2,3] at most $n^{2\cdot 3}$ many — Prob. that such a cut is bad $n^{-10\cdot 2}$
- Cuts of value [3,4] at most $n^{2\cdot 4}$ many — Prob. that such a cut is bad $n^{-10\cdot 3}$
- Cuts of value [4,5] at most $n^{2\cdot 5}$ many — Prob. that such a cut is bad $n^{-10\cdot 4}$

- Cuts of value [n-1, n] at most $n^{2\cdot n}$ many — Prob. that such a cut is bad $n^{-10\cdot(n-1)}$

By union bound,

$$\Pr[H\ is\ good] \geq 1 - \sum_{i=1}^{n-1} n^{2\cdot(i+1)-10i} = 1 - \sum_{i=1}^{n-1} \frac{n^2}{n^{8i}} \geq 1 - \frac{1}{n}$$

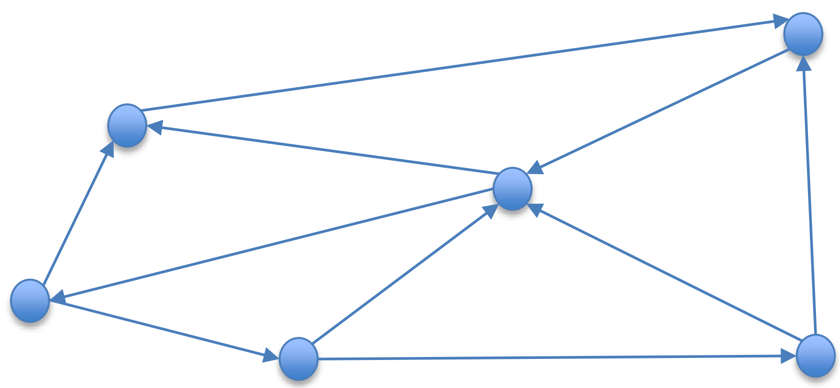
Randomized Round

Find an optimal solution x^* to LP relaxation

Scale up x^* by taking $K \coloneqq 1000\ln(n)$ parallel copies of each edge, each of same LP-value as the original edge

Form H by taking each edge with probability equal to its LP-value

Compute Eulerian graph by finding an (integral) min cost circulation with lower bound 1 for each arc in H



Well connected?

Eulerian?

So w.h.p. H is well connected and almost Eulerian

We will use these facts to bound the cost of the last step

Assuming H is good, then there exists a circulation on H where each arc has lower bound 1 and upper bound 2

Note that this implies that the cost of a min-cost circulation with lower bound 1 on each edge in H is at most two times the cost of H

VERIFY CONDITIONS FROM HOFFMAN's CIRCULATION THM:

- For every arc $e \in E$ we must have $l(e) \leq u(e)$ 
- For every $S \subset V$ we must have $l\big(\delta^-(S)\big) \leq u\big(\delta^+(S)\big)$

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For second condition,

$$l\big(\delta_H^-(S)\big) \leq \frac{4}{3} K\, x^*\big(\delta^-(S)\big)$$

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For second condition,

$$l\big(\delta_H^-(S)\big) \leq \frac{4}{3} K\, x^*\big(\delta^-(S)\big) = 2\left(1 - \frac{1}{3}\right) K\, x^*\big(\delta^+(S)\big)$$

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Expected cost of Tour is at most twice the cost of H.

- **What is the expected cost of H?**

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Okay we are interested in expected cost of H *conditioned on it being good.*

But this is $\leq \frac{K\ ln\, n}{1-1/n}$ *which between friends is* $K\ ln\, n$

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THEOREM: Goemans, Harvey, Jain, Singh'10

Randomized round returns an $O(\log n)$-approximate tour w.h.p.

Main ingredients

- O(log n) guarantee from ensuring connectivity
- Chernoff bounds ensured concentration which was useful for bounding the parity correction cost
- Karger's result allowed us to apply the union bound in a smart way

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SPANNING TREES:

- Always connected
- Negative correlation still allows for the application of Chernoff'bounds

THIN SPANNING TREES

Thin Trees

Thin Trees:

Let T be a spanning tree of $G = (V, E, w)$ and x^* an optimal LP solution.

T is α-thin (w.r.t) x^* if for every $S \subset V$

$$|\boldsymbol{\delta_T(S)}| \leq \boldsymbol{\alpha x^*(\delta^+(S))}$$

Thin Trees to Tours:

Let T be an α-thin spanning tree of $G = (V, E, w)$ and x^* an optimal LP solution.

Then there is a tour of value at most $w(T) + O(\alpha)OPT_{LP}$

Outline of proof

- The circulation network G that for each edge $e \in E$ has

$$l(e) = \begin{cases} 1, & e \in T \\ 0, & e \notin T \end{cases} \quad \text{and } u(e) = l(e) + \alpha x_e^*$$

has a feasible circulation. **Remains to prove this!**

- This circulation has cost at most $\sum_{e \in E} u(e)w(e) \leq w(T) + \alpha OPT_{LP}$

- Hence, there is an integral min-cost circulation satisfying the lower bounds of cost at most $w(T) + \alpha OPT_{LP}$

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For second condition,

$$l\big(\delta^-(S)\big) \leq \alpha x^*\big(\delta^-(S)\big)$$

because tree is α-thin

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For second condition,

$$l\big(\delta^-(S)\big) \underset{\substack{\uparrow \\ \text{because tree is } \alpha\text{-thin}}}{\leq} \alpha x^*\big(\delta^-(S)\big) = \alpha x^*\big(\delta^+(S)\big) \underset{\substack{\uparrow \\ \text{by def. of u}}}{\leq} u(\delta^+(S))$$

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Let T be an α-thin spanning tree of $G = (V, E, w)$ and x^* an optimal LP solution.

Then there is a tour of value at most $\boldsymbol{w(T) + O(\alpha)OPT_{LP}}$

METHODS FOR FINDING THIN TREES

Spanning Tree Round

Find an optimal solution x^* to LP relaxation

Let $z_{\{uv\}} = (x^*_{uv}+x^*_{vu}) \cdot \left(1 - \frac{1}{n}\right)$ be a feasible point to the spanning tree polytope and sample a spanning tree T with negative correlation satisfying these marginals

Compute Eulerian graph by finding an (integral) min cost circulation with lower bound 1 for each arc in spanning tree

- What is the expected cost of the spanning tree? $\left(1 - \frac{1}{n}\right) OPT_{LP}$
- How thin is the tree? We can apply upper Chernoff bound:

For any $S \subset V$,

$$\Pr\left[|\delta_T(S)| > 1000 \frac{\log n}{\log \log n} x^*(\delta^+(S))\right] < \frac{1}{n^{10x^*(\delta^+(S))}}$$

This together with Karger implies that the tree is w.h.p $O\left(\frac{\log n}{\log \log n}\right)$-thin

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Theorem: Asadpour, Goemans, Madry, Oveis Gharan, Saberi'10

Spanning tree algorithm is a $\boldsymbol{O\left(\frac{\log n}{\log\log n}\right)}$-approximation algorithm for ATSP

$$\Pr[|\delta_T(S)| > 1000\,\frac{\log n}{\log\log n}\,x^*(\delta^+(S))] < \frac{1}{n^{10x^*(\delta^+(S))}}$$

This together with Karger implies that the tree is w.h.p $\boldsymbol{O\left(\frac{\log n}{\log\log n}\right)}$-thin

State of the Art of Thin Tree Approach

Theorem: Asadpour, Goemans, Madry, Oveis Gharan, Saberi'10

A randomized polytime algorithm gives a $\boldsymbol{O}\left(\frac{\boldsymbol{\log n}}{\boldsymbol{\log\log n}}\right)$-thin tree

Theorem: Anari, Oveis Gharan'14

There exists a $\boldsymbol{O(polyloglog\ n)}$-thin tree

These results imply a $O\left(\frac{\log n}{\log\log n}\right)$-approximation algorithm and a $O(poly\log\log n)$ bound on the integrality gap

Open Problem: Is there always a $O(1)$-thin tree?

Yes for graphs of bounded genus [Oveis Gharan and Saberi'11]