Lecture 3: Asymmetric TSP (cont)

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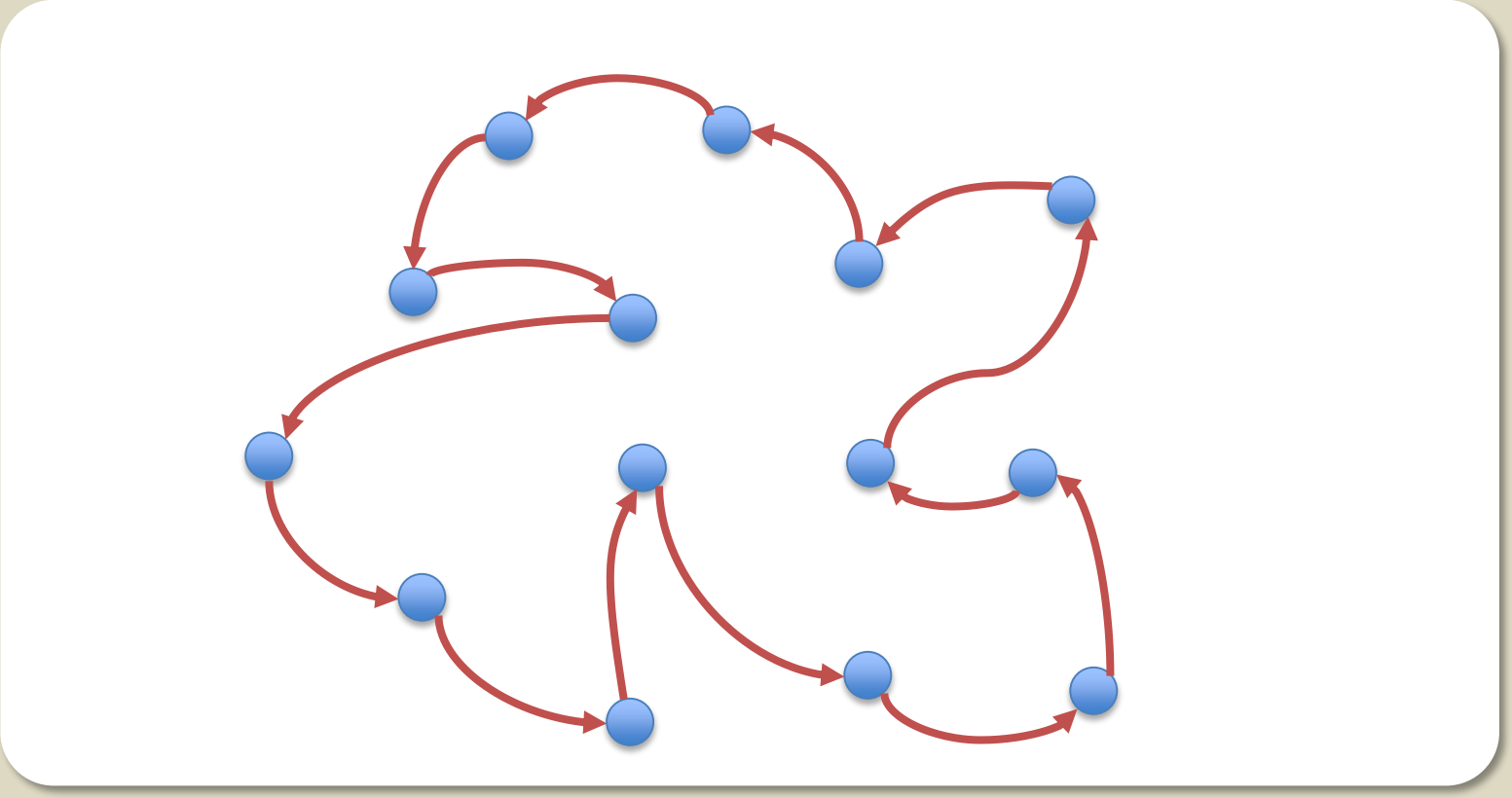
Asymmetric Traveling Salesman Problem

INPUT: a complete digraph $G = (V, E)$ with pairwise (*not necessarily symmetric*) distances that satisfy the triangle inequality

OUTPUT: a tour of minimum weight that visits each vertex once



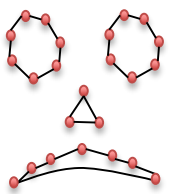
Held-Karp Relaxation of ATSP

Variables: x_{uv} = "indicate whether arc (u, v) is used in tour"

Held-Karp Relaxation

Minimize: $\sum_{uv \in E} w(u, v)\, x_{uv}$

Subject to: $\boldsymbol{x(\delta^+(v)) = x(\delta^-(v)) = 1}$ for all $v \in V$



$\boldsymbol{x(\delta^+(S)) \geq 1}$ for all $S \subset V$

$x \geq 0$

TOOLS FOR ROUNDING LP

Circulations

INPUT: a digraph $G = (V, E)$ and for each arc $e \in E$

a lower bound $l(e) \geq 0$ and an upper bound $u(e) \geq 0$

OUTPUT: a circulation $f: E \to R_+$ satisfying

flow conservation: $f(\delta^+(v)) = f(\delta^-(v))$ for each $v \in V$

edge bounds: $l(e) \leq f(e) \leq u(e)$ for each $e \in E$

Blue edges have lower bound 1 and upper bound 2

Red edges have lower bound 0 and upper bound ∞

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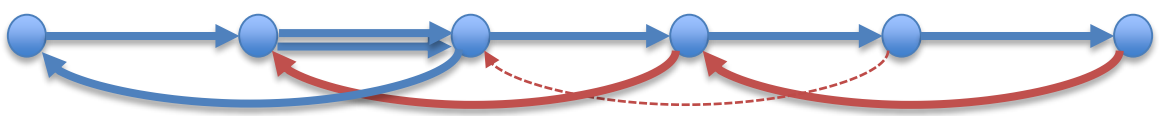


Blue edges have lower bound 1 and upper bound 2

Red edges have lower bound 0 and upper bound ∞

Can be calculated in polytime and also a min cost circulation can be found

When does a circulation exist?

NECESSARY CONDITIONS:

- For every arc $e \in E$ we must have $l(e) \leq u(e)$
- For every $S \subset V$ we must have $l\big(\delta^-(S)\big) \leq u\big(\delta^+(S)\big)$

HOFFMAN's CIRCULATION THEOREM: Hoffman'60

The above conditions are also **sufficient.**

Furthermore, if l and u are integer valued, the circulation f can be chosen to be integral.

Basic idea select a subset of edges and make it Eulerian by finding a circulation

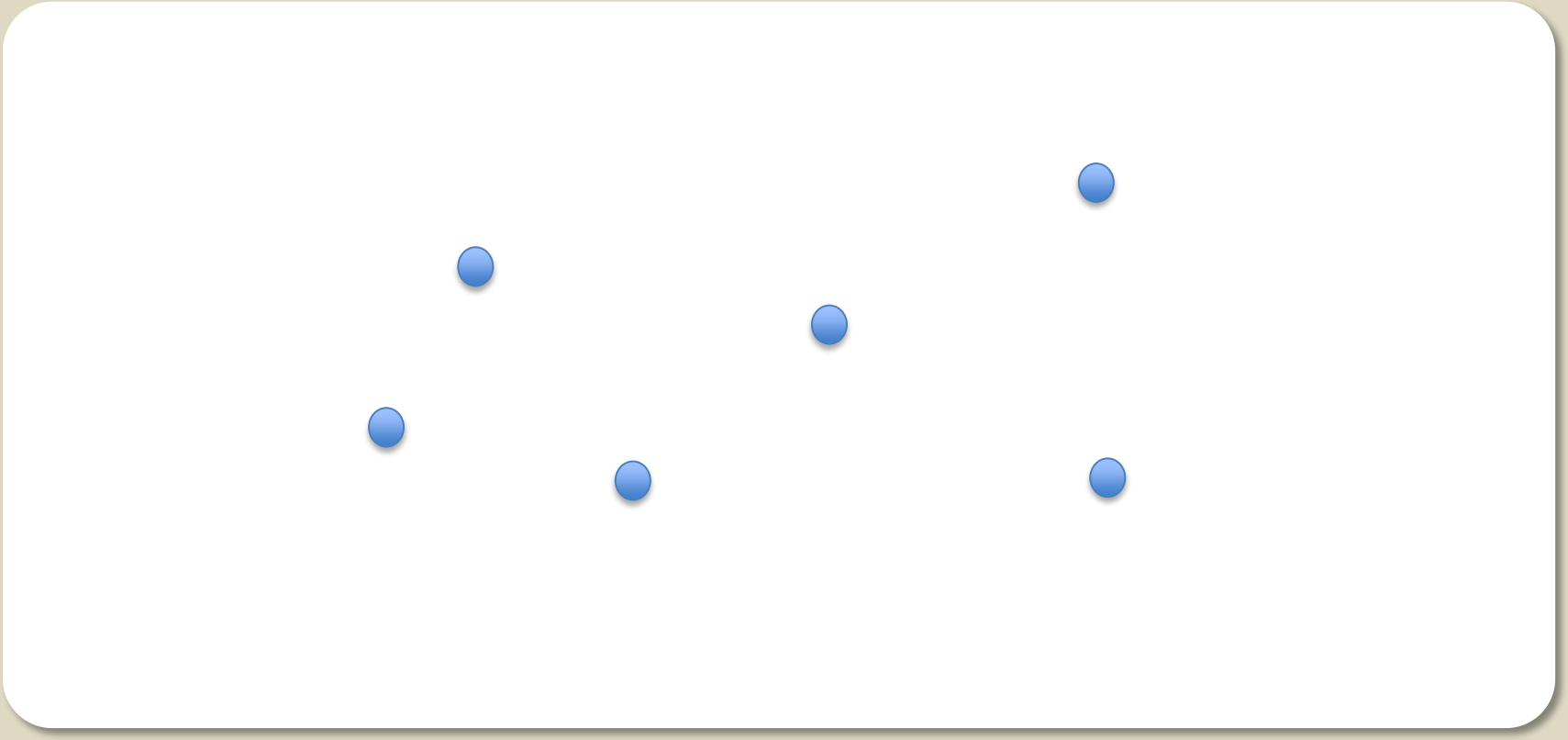
RANDOMIZED LP ROUNDING

Randomized Round

Find an optimal solution x^* to LP relaxation

Form H by taking each edge with probability equal to its LP-value

Compute Eulerian graph by finding an (integral) min cost circulation with lower bound 1 for each arc in H



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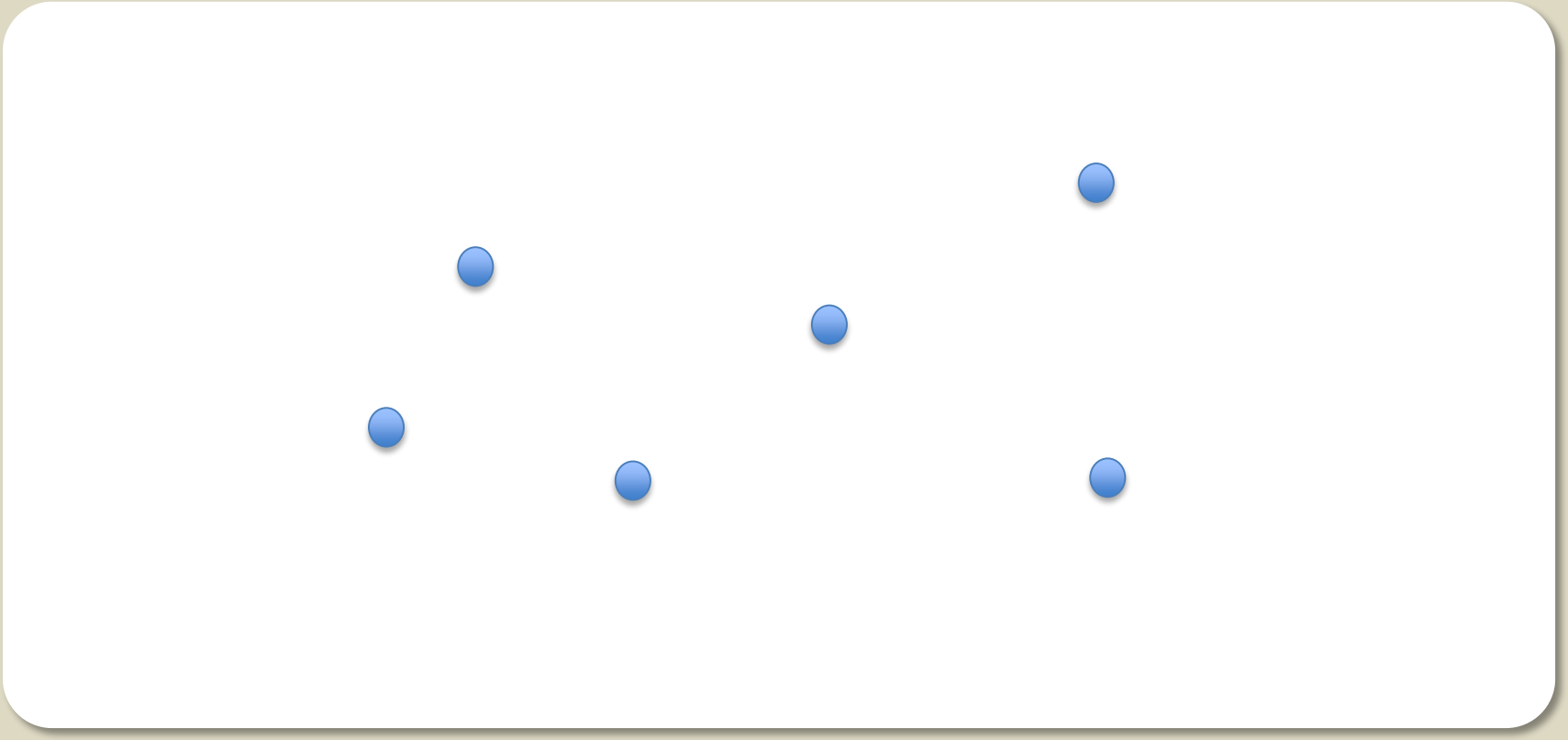

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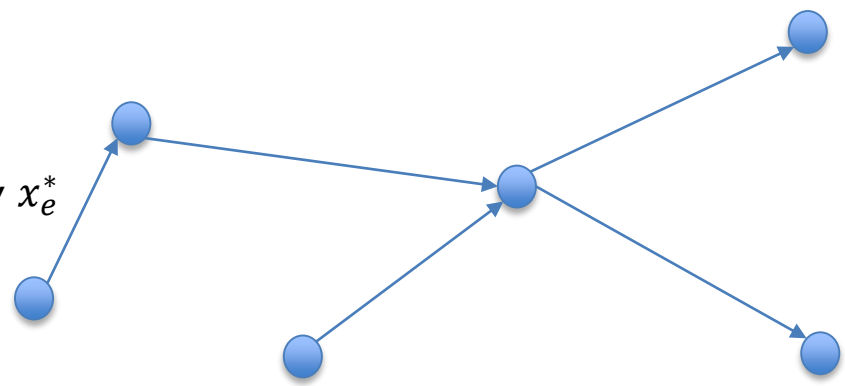
Taken with probability x_e^*



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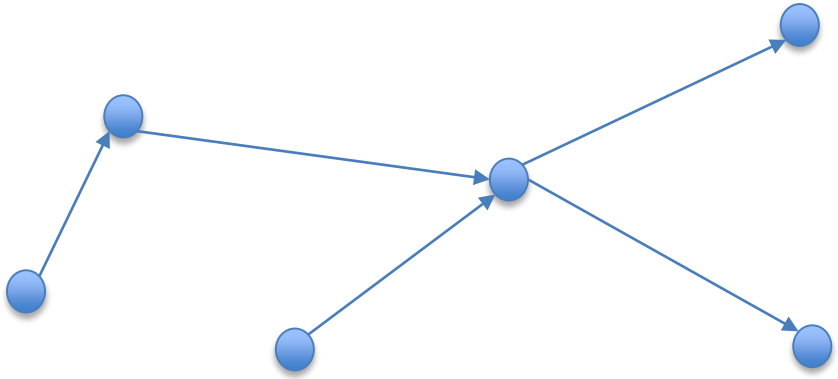

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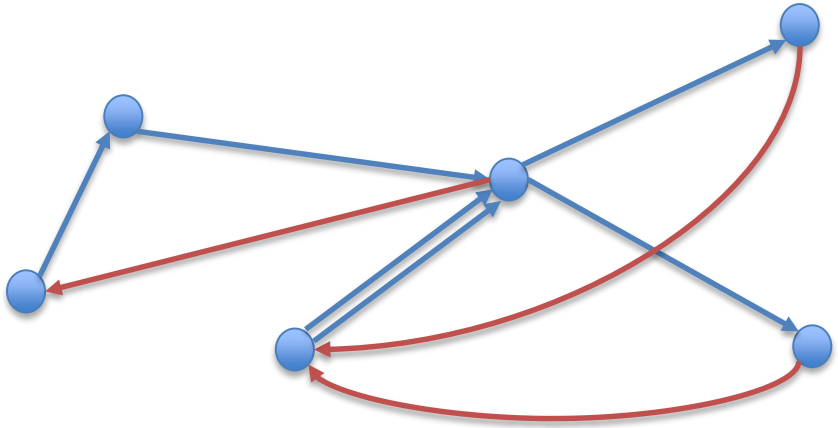

What's the problem?

- With high probability the sampled graph H is not even connected

- So we will return an Eulerian graph but it is **not** connected ☹

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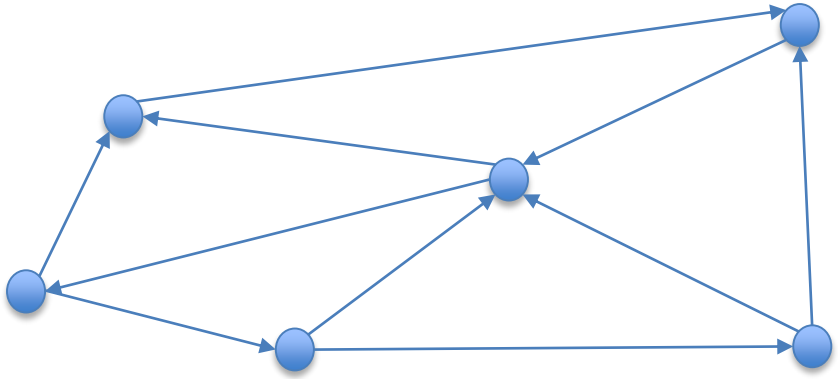


Well connected?

Eulerian?

Analyzing H: out-degree of a vertex

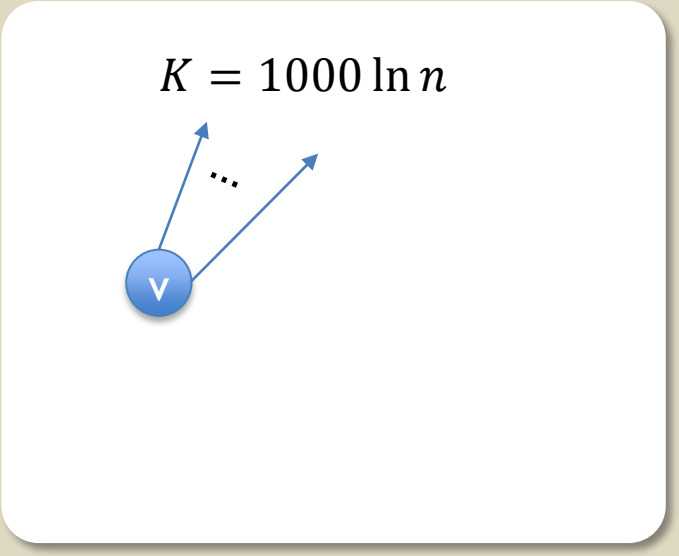
What is the expected out-degree of v?

It is $K \cdot x^*\big(\delta^+(v)\big) = 1000 \ln n =: \mu$

The number of outgoing edges is the sum of random independent 0/1 variables

Hence, by standard Chernoff bound

$$\Pr\left[\left|\left|\delta_H^+(v)\right| - \mu\right| \geq \frac{\mu}{3}\right] \leq e^{-\frac{\mu}{30}} \leq \frac{1}{2n^{10x^*(\delta^+(v))}}$$



In words: the number of edges will deviate

from its expectation more than a fraction 1/3 with probability at most $O(\frac{1}{n^{10}})$

Analyzing H: in-degree of a vertex

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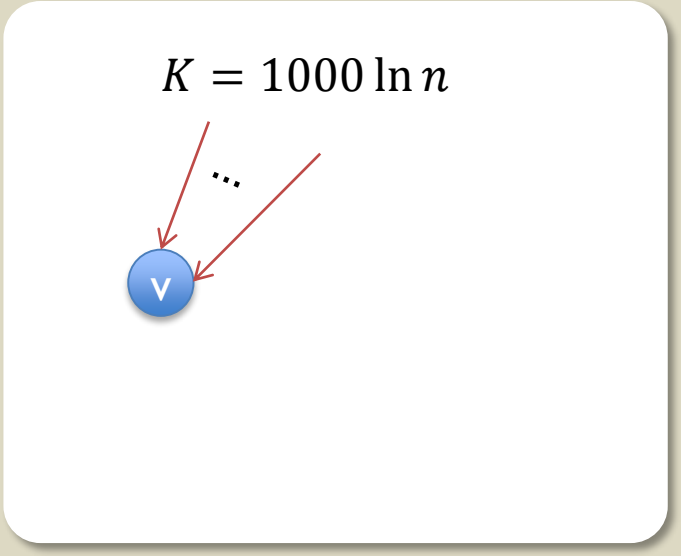

In words: the number of edges will deviate

from its expectation more than a fraction 1/3 with probability at most $O(\frac{1}{n^{10}})$

Analyzing H: in/out-degree of a set S

What is the expected in-degree of S?

It is $K \cdot x^*\big(\delta^-(S)\big) = 1000 \ln n \; x^*(\delta^-(S)) =: \mu$

The number of outgoing edges is the sum of random independent 0/1 variables

Hence, by standard Chernoff bound

$$\Pr\left[\left||\delta_H^-(S)| - \mu\right| \geq \frac{\mu}{3}\right] \leq e^{-\frac{\mu}{30}} \leq \frac{1}{2n^{10x^*(\delta^-(S))}}$$

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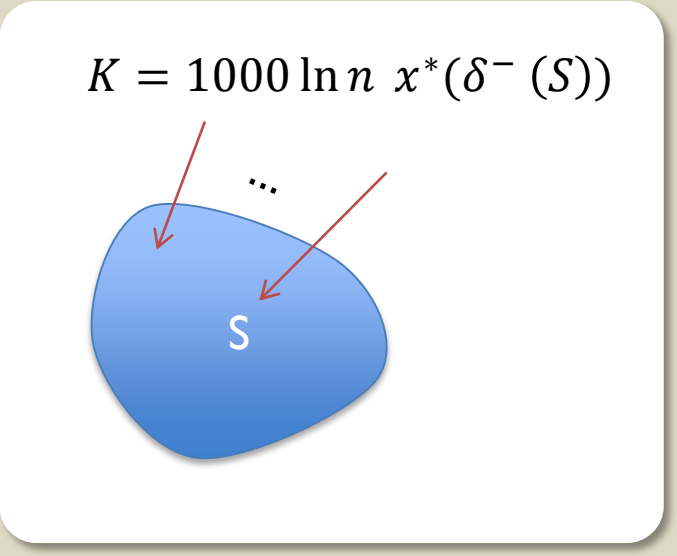

Analyzing H: bad cuts

We say that a cut S is bad in H if the incoming and outgoing edges deviate more than a fraction1/3

S is bad in H if

$$\left||\delta_H^-(S)| - \mu\right| \geq \frac{\mu}{3} \quad \text{or} \quad \left||\delta_H^+(S)| - \mu\right| \geq \frac{\mu}{3}$$

where $\mu = x^*\left(\delta^-(S)\right) = x^*(\delta^+(S))$

By previous calculations

$$\Pr[S\ is\ bad] \leq \Pr\left[\left||\delta_H^-(v)| - \mu\right| \geq \frac{\mu}{3}\right] + \Pr\left[\left||\delta_H^+(v)| - \mu\right| \geq \frac{\mu}{3}\right]$$

$$\leq \frac{1}{n^{10x^*(\delta^+(S))}}$$

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By previous calculations

$$\mathbf{Pr}[\boldsymbol{S\ is\ bad}]$$

$$\leq \frac{\mathbf{1}}{\boldsymbol{n}^{\mathbf{10}\boldsymbol{x}^*(\boldsymbol{\delta}^+(\boldsymbol{S}))}}$$

Probability that H is good?

H is good if no cut is bad

We know that a single cut S is bad w.p $\leq \frac{1}{n^{10x^*\left(\delta^+(S)\right)}} \leq \frac{1}{n^{10}}$

But 2^n many cuts so we can't make union bound…

Remedy: beautiful result by Karger

BOUNDING NUMBER OF SMALL CUTS: Karger

Consider an undirected graph $G = (V, E)$ with edge-weights w. Let c be the value of a min-cut.

Then the number of cuts of value αc is at most $\leq n^{2\alpha}$

But our graph is directed, **why can we still use the above theorem?**

It is Eulerian (w.r.t. to weights x^*) so any cut of out-degree/in-degree c corresponds to a cut of value 2c in the undirected graph

Probability that H is good?

- Cuts of value [1,2] at most $n^{2\cdot 2}$ many Prob. that such a cut is bad $n^{-10\cdot 1}$
- Cuts of value [2,3] at most $n^{2\cdot 3}$ many Prob. that such a cut is bad $n^{-10\cdot 2}$
- Cuts of value [3,4] at most $n^{2\cdot 4}$ many Prob. that such a cut is bad $n^{-10\cdot 3}$
- Cuts of value [4,5] at most $n^{2\cdot 5}$ many Prob. that such a cut is bad $n^{-10\cdot 4}$

- Cuts of value [n-1, n] at most $n^{2\cdot n}$ many Prob. that such a cut is bad $n^{-10\cdot(n-1)}$

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By union bound,

$$\Pr[H\ is\ good] \geq 1 - \sum_{i=1}^{n-1} n^{2\cdot(i+1)-10i}$$

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Scale up x^* by taking $K := 1000\ln(n)$ parallel copies of each edge, each of same LP-value as the original edge

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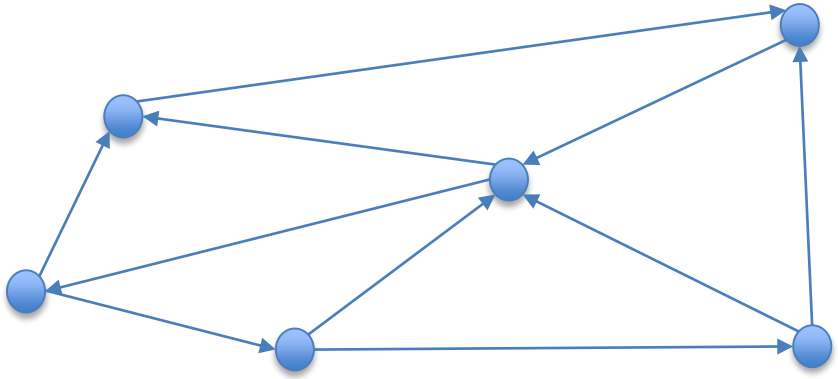

Well connected?

Eulerian?

So w.h.p. H is well connected and almost Eulerian

We will use these facts to bound the cost of the last step

Assuming H is good, then there exists a circulation on H where each arc has lower bound 1 and upper bound 2

Note that this implies that the cost of a min-cost circulation with lower bound 1 on each edge in H is at most two times the cost of H

VERIFY CONDITIONS FROM HOFFMAN's CIRCULATION THM:

- For every arc $e \in E$ we must have $l(e) \leq u(e)$ 
- For every $S \subset V$ we must have $l\left(\delta^{-}(S)\right) \leq u\left(\delta^{+}(S)\right)$

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For second condition,

$$l\big(\delta_H^-(S)\big) \leq \frac{4}{3} K\, x^*\big(\delta^-(S)\big)$$

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$$l\big(\delta_H^-(S)\big) \leq \frac{4}{3} K\, x^*\big(\delta^-(S)\big) = 2\left(1 - \frac{1}{3}\right) K\, x^*\big(\delta^+(S)\big)$$

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Compute Eulerian graph by finding an (integral) min cost circulation with lower bound 1 for each arc in H

Expected cost of Tour is at most twice the cost of H.

- **What is the expected cost of H?**

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- **What is the expected cost of H?** K times the LP cost

Okay we are interested in expected cost of H conditioned on it being good.

But this is $\leq \frac{K}{1-1/n}$ *which between friends is* $K \ln n$

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But this is $\leq \frac{K}{1-1/n}$ *which between friends is* K

THEOREM: Goemans, Harvey, Jain, Singh'10

Randomized round returns an $O(\log n)$-approximate tour w.h.p.

Main ingredients

- O(log n) guarantee from ensuring connectivity
- Chernoff bounds ensured concentration which was useful for bounding the parity correction cost
- Karger's result allowed us to apply the union bound in a smart way

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SPANNING TREES:

- Always connected
- Negative correlation still allows for the application of Chernoff'bounds

THIN SPANNING TREES

Thin Trees

Thin Trees:

Let T be a spanning tree of $G = (V, E, w)$ and x^* an optimal LP solution.

T is α-thin (w.r.t) x^* if for every $S \subset V$

$$|\delta_T(S)| \leq \alpha x^*(\delta^+(S))$$

Thin Trees to Tours:

Let T be an α-thin spanning tree of $G = (V, E, w)$ and x^* an optimal LP solution.

Then there is a tour of value at most $w(T) + O(\alpha) OPT_{LP}$

Outline of proof

- The circulation network G that for each edge $e \in E$ has

$$l(e) = \begin{cases} 1, & e \in T \\ 0, & e \notin T \end{cases} \qquad \text{and } u(e) = l(e) + \alpha x_e^*$$

has a feasible circulation. **Remains to prove this!**

- This circulation has cost at most $\sum_{e \in E} u(e)w(e) \leq w(T) + \alpha OPT_{LP}$

- Hence, there is an integral min-cost circulation satisfying the lower bounds of cost at most $w(T) + \alpha OPT_{LP}$

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For second condition,

$$l\big(\delta^-(S)\big) \leq \alpha x^*\big(\delta^-(S)\big)$$

because tree is α-thin

The circulation network G that for each edge $e \in E$ has

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For second condition,

$$l(\delta^-(S)) \underset{\substack{\uparrow \\ \text{because tree is } \alpha\text{-thin}}}{\leq} \alpha x^*(\delta^-(S)) = \alpha x^*(\delta^+(S)) \underset{\substack{\uparrow \\ \text{by def. of u}}}{\leq} u(\delta^+(S))$$

Thin Trees to Tours:

Let T be an α-thin spanning tree of $G = (V, E, w)$ and x^* an optimal LP solution.

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METHODS FOR FINDING THIN TREES

Spanning Tree Round

Find an optimal solution x^* to LP relaxation

Let $z_{\{uv\}} = (x^*_{uv} + x^*_{vu}) \cdot \left(1 - \frac{1}{n}\right)$ be a feasible point to the spanning tree polytope and sample a spanning tree T with negative correlation satisfying these marginals

Compute Eulerian graph by finding an (integral) min cost circulation with lower bound 1 for each arc in spanning tree

- What is the expected cost of the spanning tree? $\left(1 - \frac{1}{n}\right) OPT_{LP}$

- How thin is the tree? We can apply upper Chernoff bound:

For any $S \subset V$,

$$\Pr[|\delta_T(S)| > 1000 \frac{\log n}{\log \log n} x^*(\delta^+(S))] < \frac{1}{n^{10x^*(\delta^+(S))}}$$

This together with Karger implies that the tree is w.h.p $O\left(\frac{\log n}{\log \log n}\right)$-thin

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Theorem: Asadpour, Goemans, Madry, Oveis Gharan, Saberi'10

Spanning tree algorithm is a $\boldsymbol{O\left(\frac{\log n}{\log \log n}\right)}$-approximation algorithm for ATSP

$$\Pr[|\delta_T(S)| > 1000 \frac{\log n}{\log \log n} x^*(\delta^+(S))] < \frac{1}{n^{10x^*(\delta^+(S))}}$$

This together with Karger implies that the tree is w.h.p $\boldsymbol{O\left(\frac{\log n}{\log \log n}\right)}$-thin

State of the Art of Thin Tree Approach

Theorem: Asadpour, Goemans, Madry, Oveis Gharan, Saberi'10

A randomized polytime algorithm gives a $\boldsymbol{O\left(\frac{\log n}{\log\log n}\right)}$-thin tree

Theorem: Anari, Oveis Gharan'14

There exists a $\boldsymbol{O(\mathrm{polyloglog}\, n)}$-thin tree

These results imply a $O\left(\frac{\log n}{\log\log n}\right)$-approximation algorithm and a

$O(poly \log\log n)$ bound on the integrality gap

Two Approaches

Easy to Find Eulerian graph

Repeatedly find cycle-covers to get $\log_2 n$-approximation

[Frieze, Galbiati, Maffiolo'82]

$0.99 \log_2 n$-approximation

[Bläser'03]

$0.84 \log_2 n$-approximation

[Kaplan, Lewenstein, Shafrir, Sviridenko'05]

$0.67 \log_2 n$-approximation

[Feige, Singh'07]

Easy to Find Connected Graph

$O(\log n / \log\log n)$-approximation

[Asadpour, Goemans, Madry, Oveis Gharan, Saberi'10]

$O(1)$-approximation for planar and bounded genus graphs

[Oveis Gharan, Saberi'11]

O($poly\ log \log n$) bound on integrality gap (generalization of Kadison-Singer)

[Anari, Oveis Gharan'14]

To summarize the two approaches

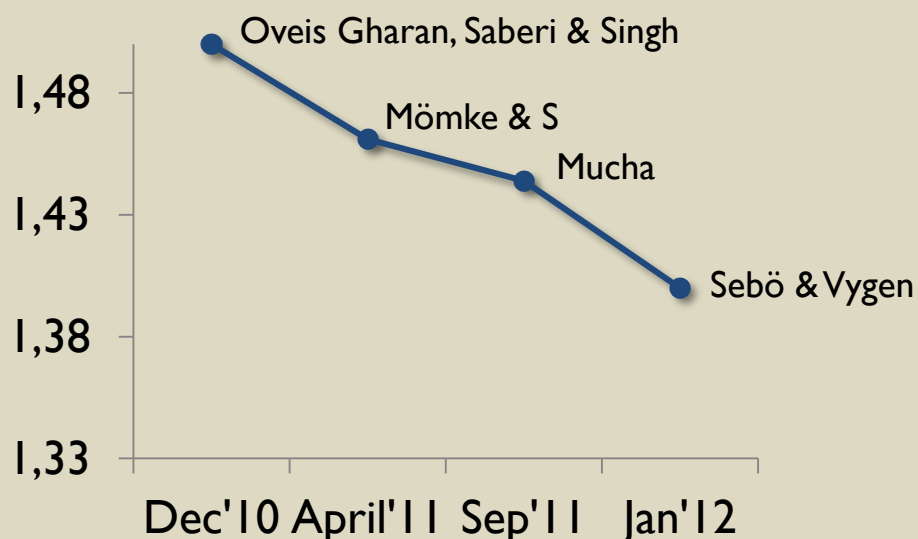
Best approximation algorithm: $O(\frac{\log n}{\log \log n})$

Best upper bound on integrality gap: $O(poly \log \log n)$

Best lower bound on integrality gap: 2

This is believed to be close to the truth

No better guarantees for shortest path metrics on unweighted graphs for which there was recent improvements for the symmetric TSP



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- Main difficulty in cycle cover approach is to bound #iterations
- In thin-tree approach we reduce ATSP to an unweighted problem

Find Eulerian graph with some connectivity requirements

A NEW APPROACH

Relaxing Connectivity

Instead of

$$x(\delta^+(v)) = x(\delta^-(v)) \qquad \text{for all } v \in V$$

$$x(\delta^+(S)) \geq 1 \qquad \textbf{for all } S \subset V$$

$$x \geq 0$$

Do for smart C

$$x(\delta^+(v)) = x(\delta^-(v)) \qquad \text{for all } v \in V$$

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Application of approach

THEOREM:

For ATSP on node-weighted graphs, the integrality gap is at most 15.

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Exist $f: V \to R^+$ s.t

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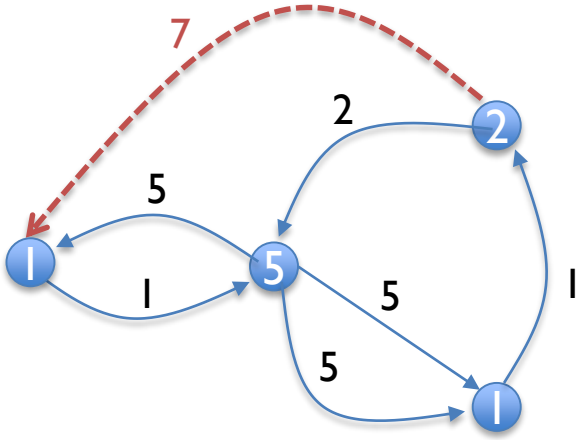

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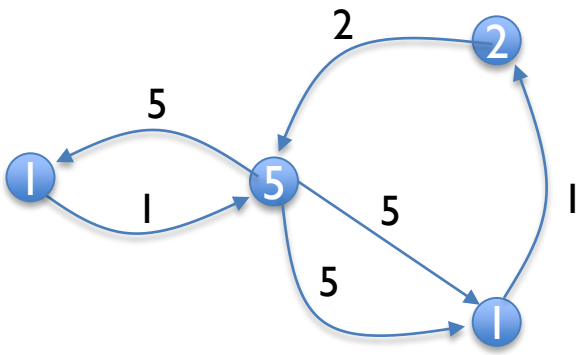

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Generalizes shortest path metrics considered for symmetric TSP

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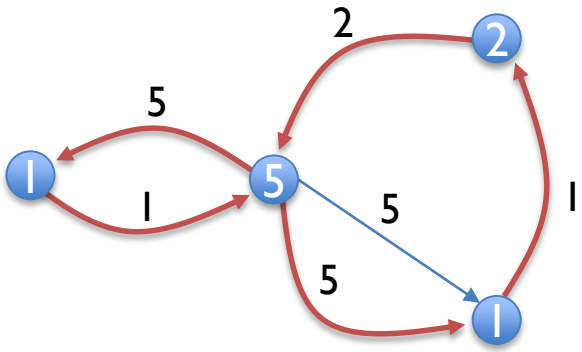

Value = 2+5+1+5+1=14

Cousin to repeated cycle cover approach

Distant relative to thin tree approach

OUR APPROACH

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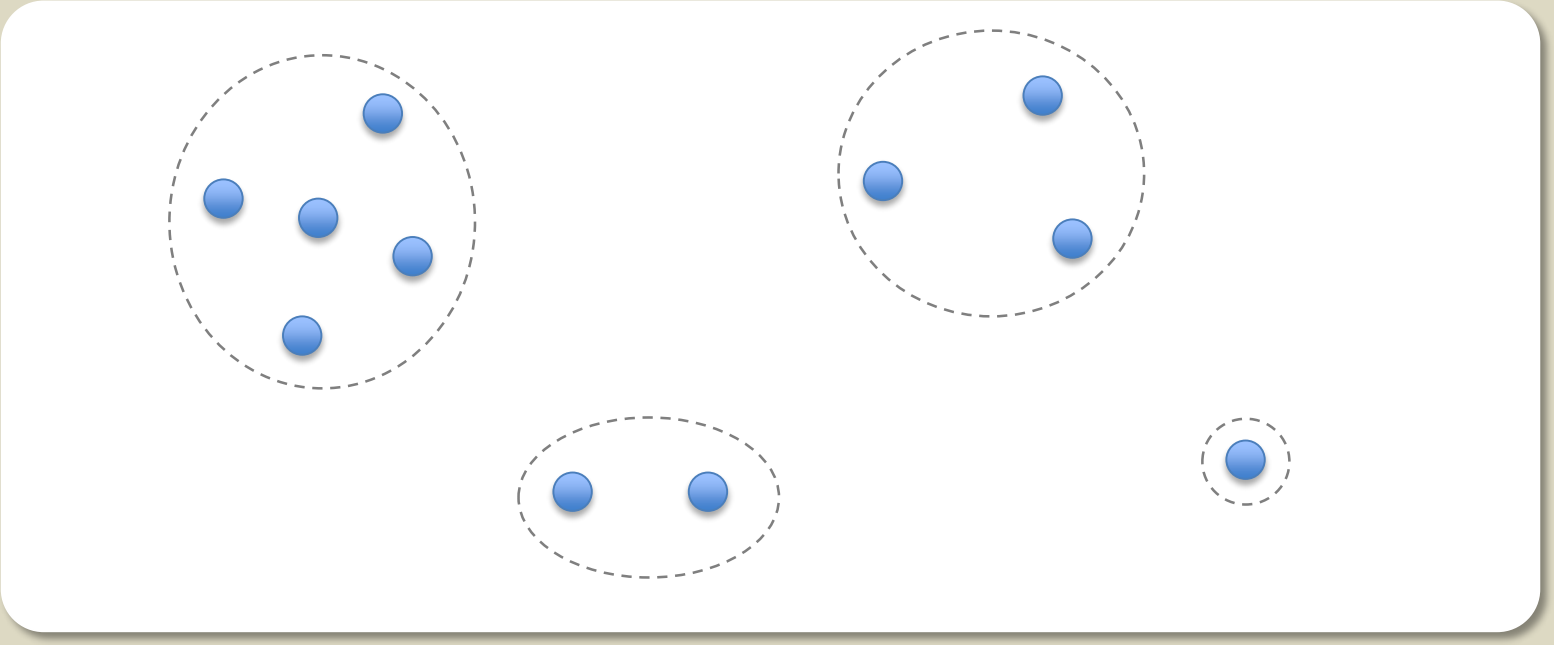
Local Connectivity ATSP

INPUT: an edge-weighted digraph $G = (V, E, w)$, a partition $V_1 \cup \cdots \cup V_k$ of V



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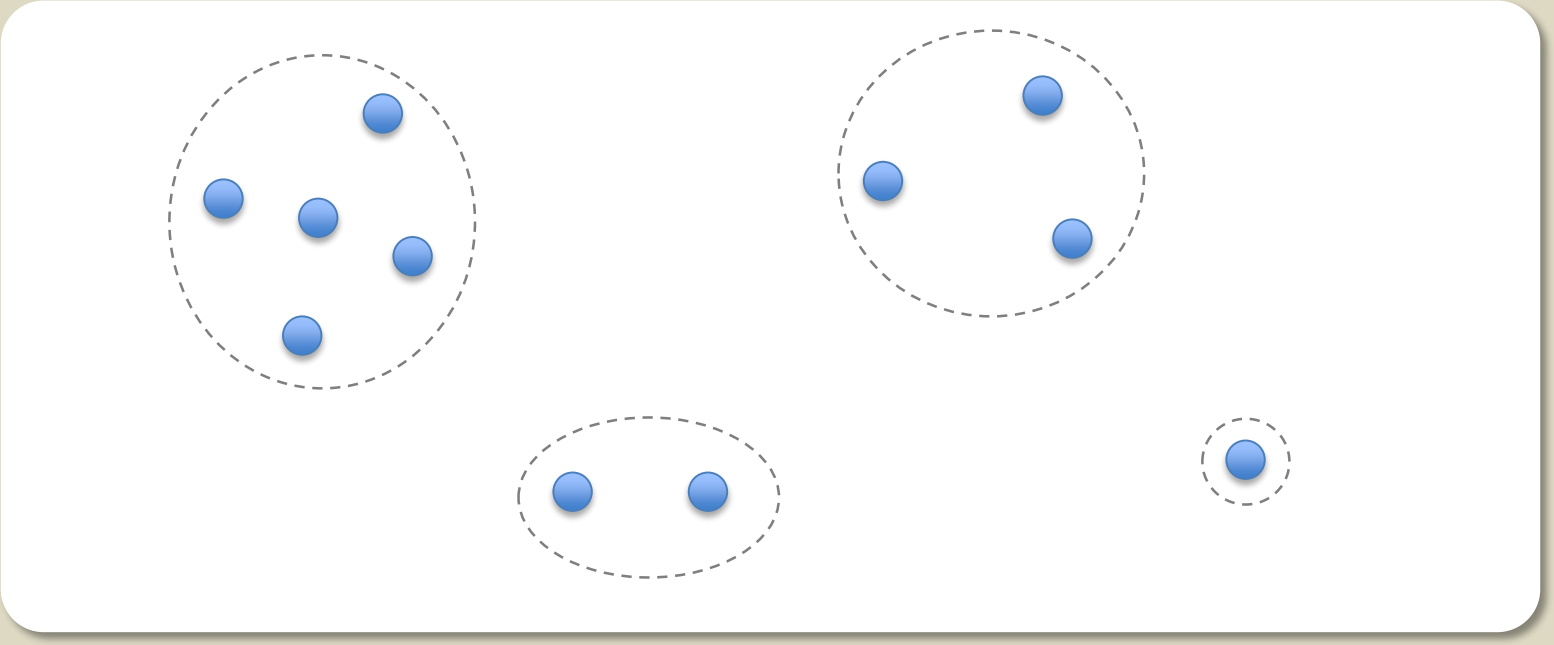

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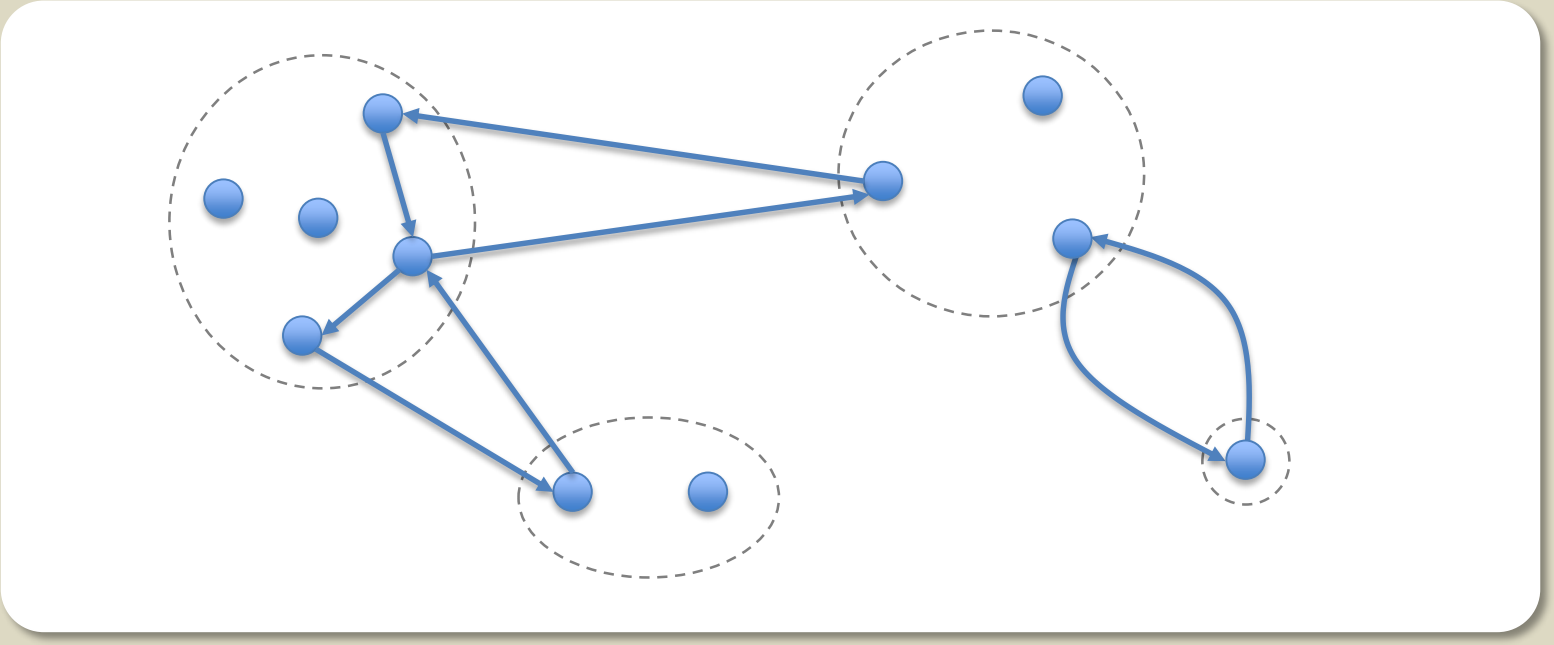
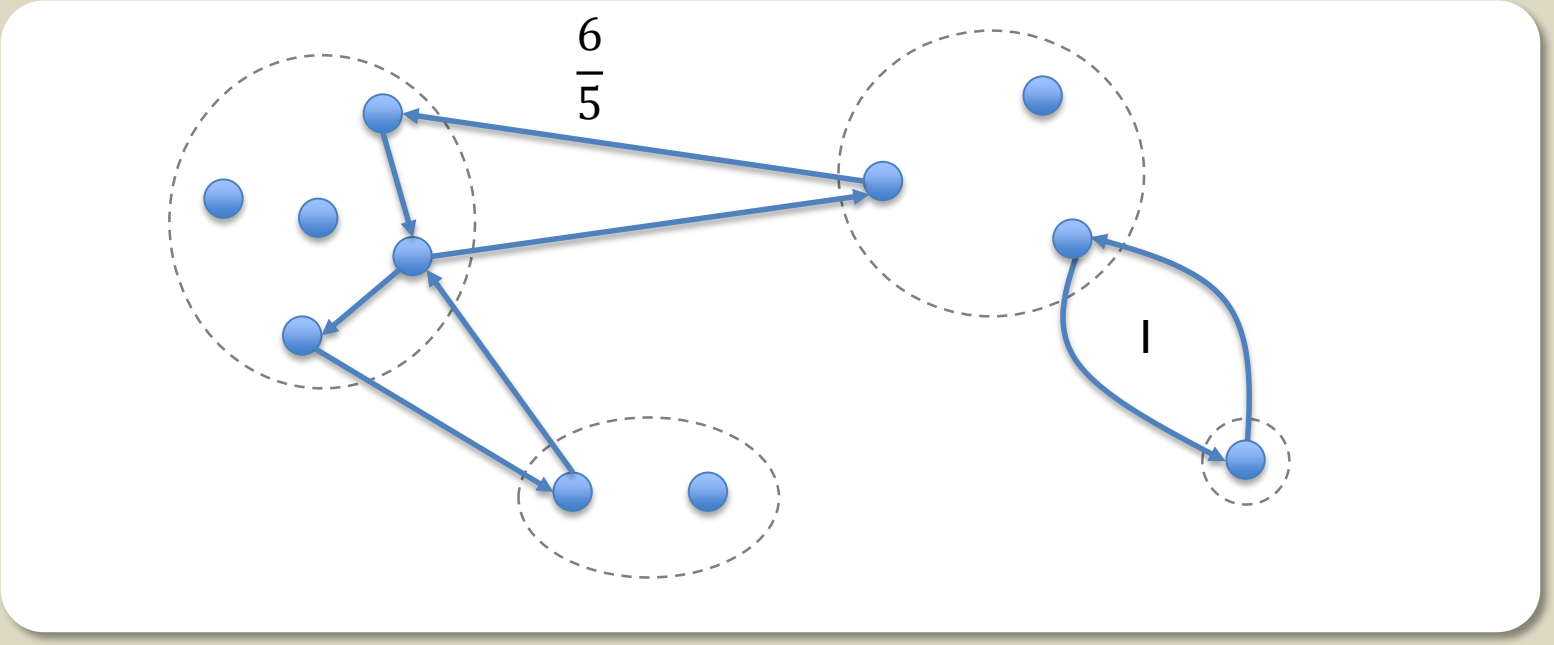


Local Connectivity ATSP

An algorithm is $\boldsymbol{\alpha}$**-light** if it always outputs a solution F s.t.

for each component in (V, F), $\frac{\#edges}{\#vertices} \leq \alpha$,

Note: designing an α-light algorithm is "easier" than an α-approximation for ATSP



Our main technical result

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If there is an α-light algorithm **A** for Local-Connectivity ATSP then the integrality gap for ATSP is at most 5α.

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If there is an α-light algorithm **A** for Local-Connectivity ATSP then the integrality gap for ATSP is at most 5α.

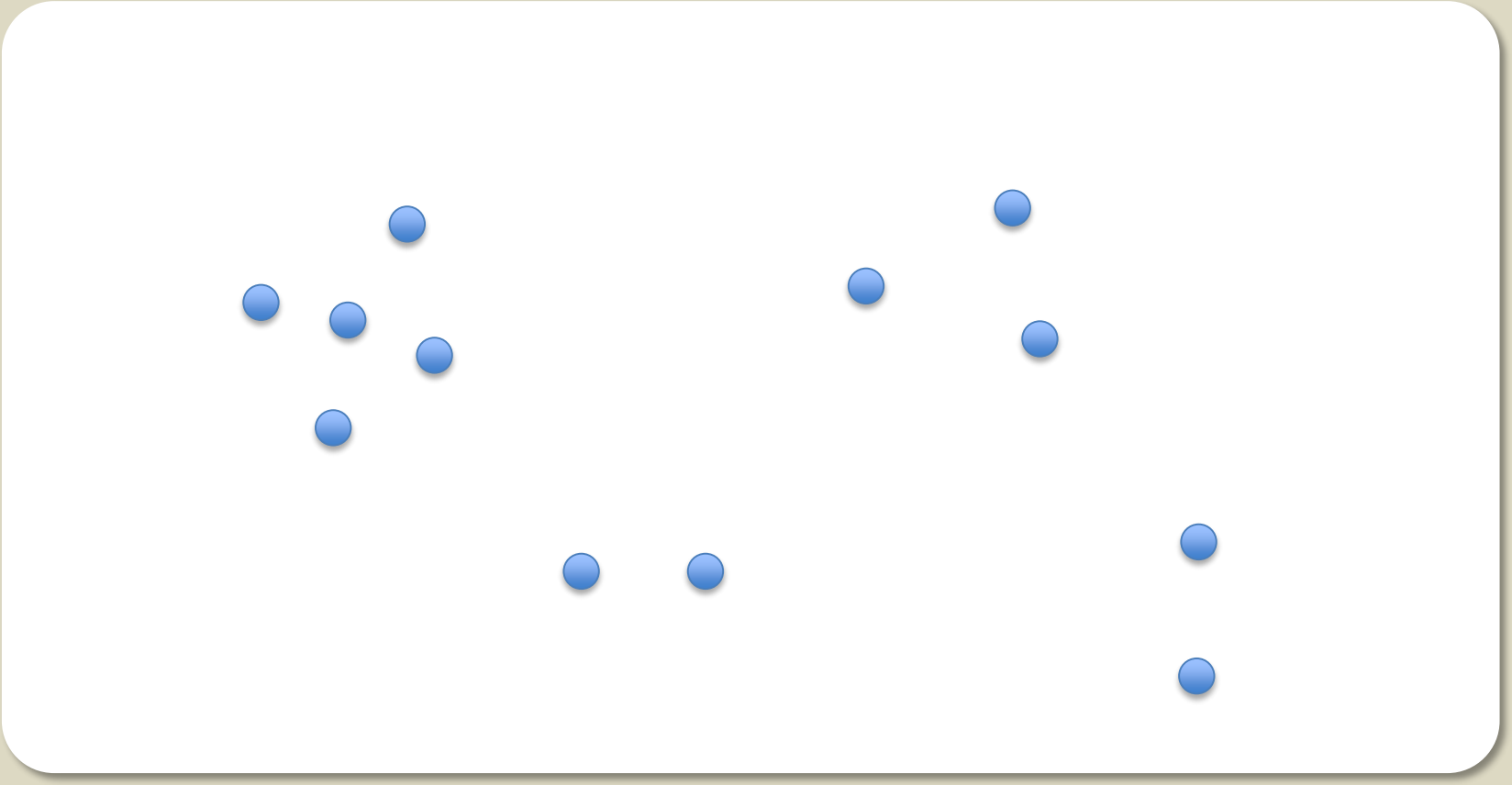
Moreover, a 9α-approximate tour can be found in polynomial time if **A** runs in polynomial time.

The problems are equivalent up to a small constant factor

There is an easy 3-light algorithm for node-weighted metric
(only part where special metric is used)

PROOF IDEA OF MAIN THM

Repeatedly solve Local-Connectivity ATSP with the current connected subgraphs as partitions



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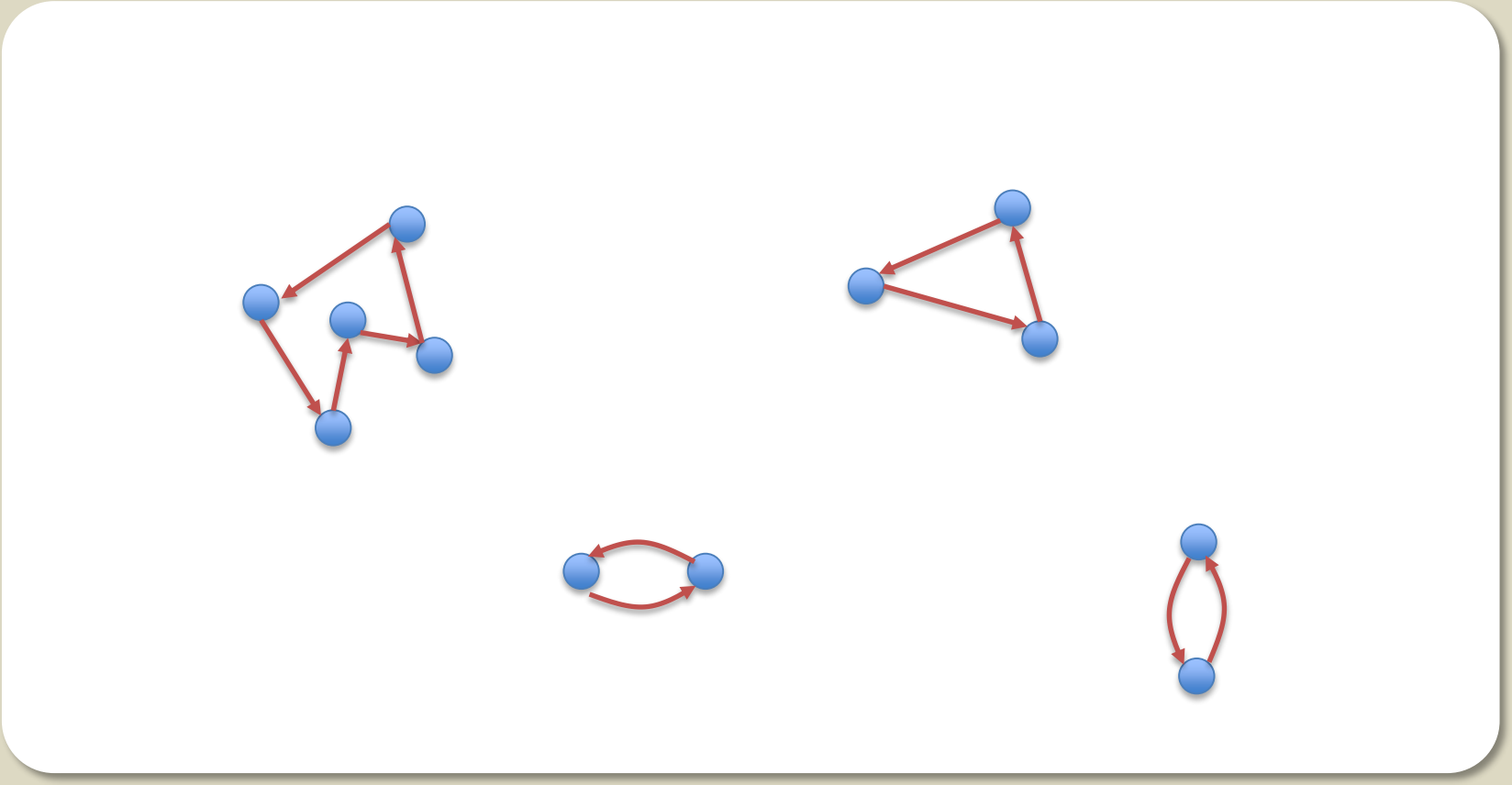
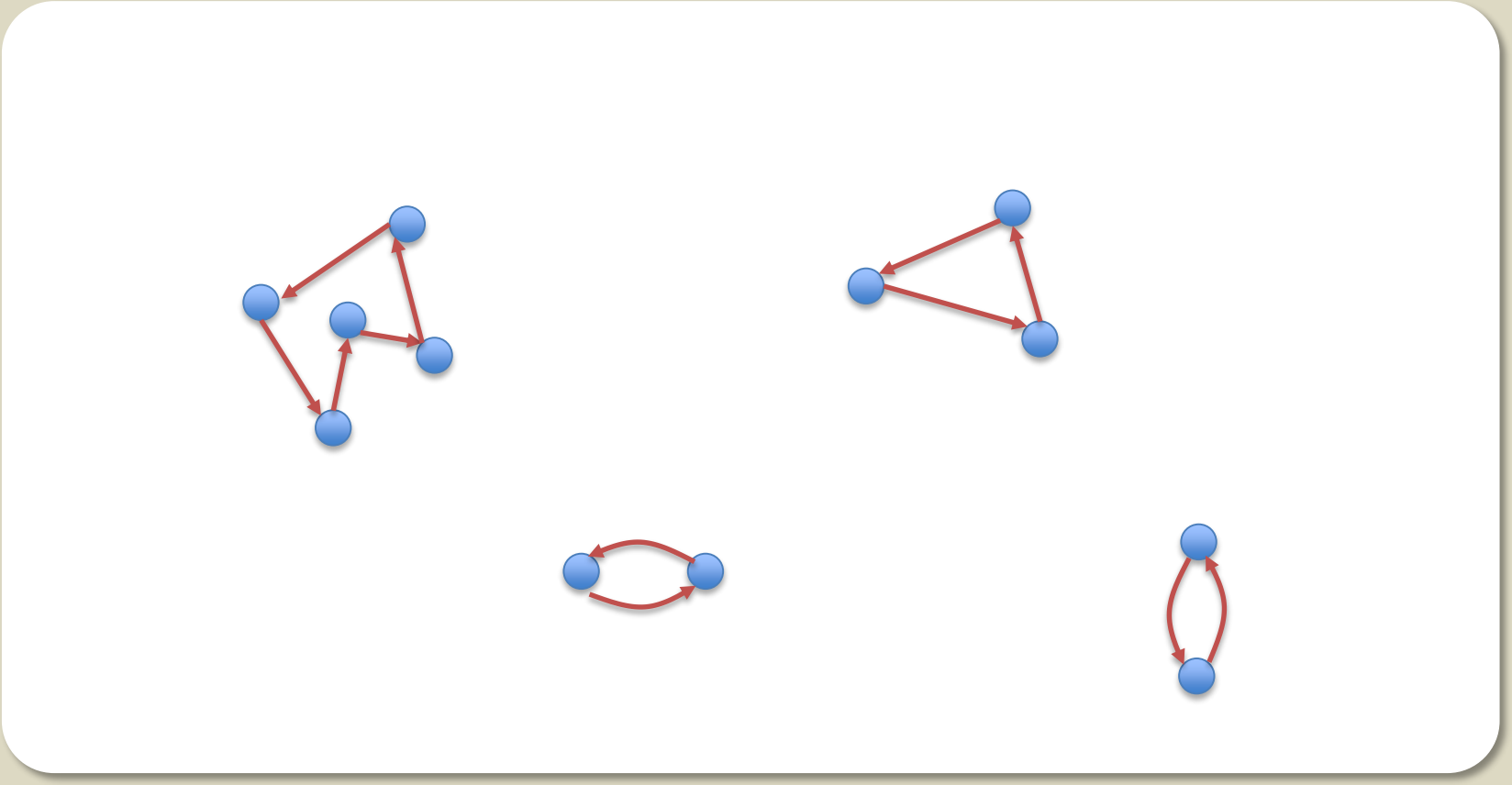


Repeatedly solve Local-Connectivity ATSP with the current connected subgraphs as partitions

Cost of red edges $\leq \alpha OPT$



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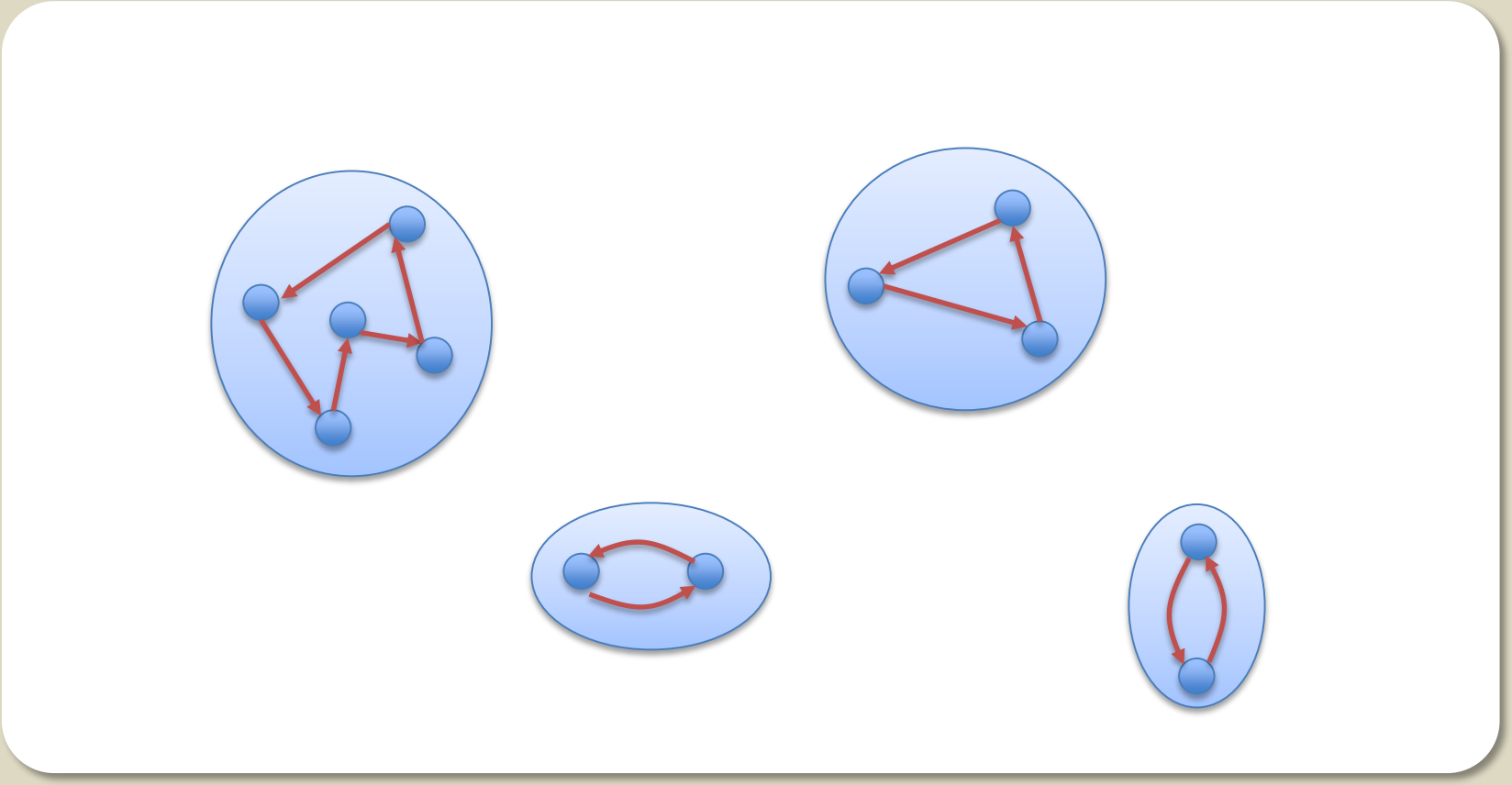

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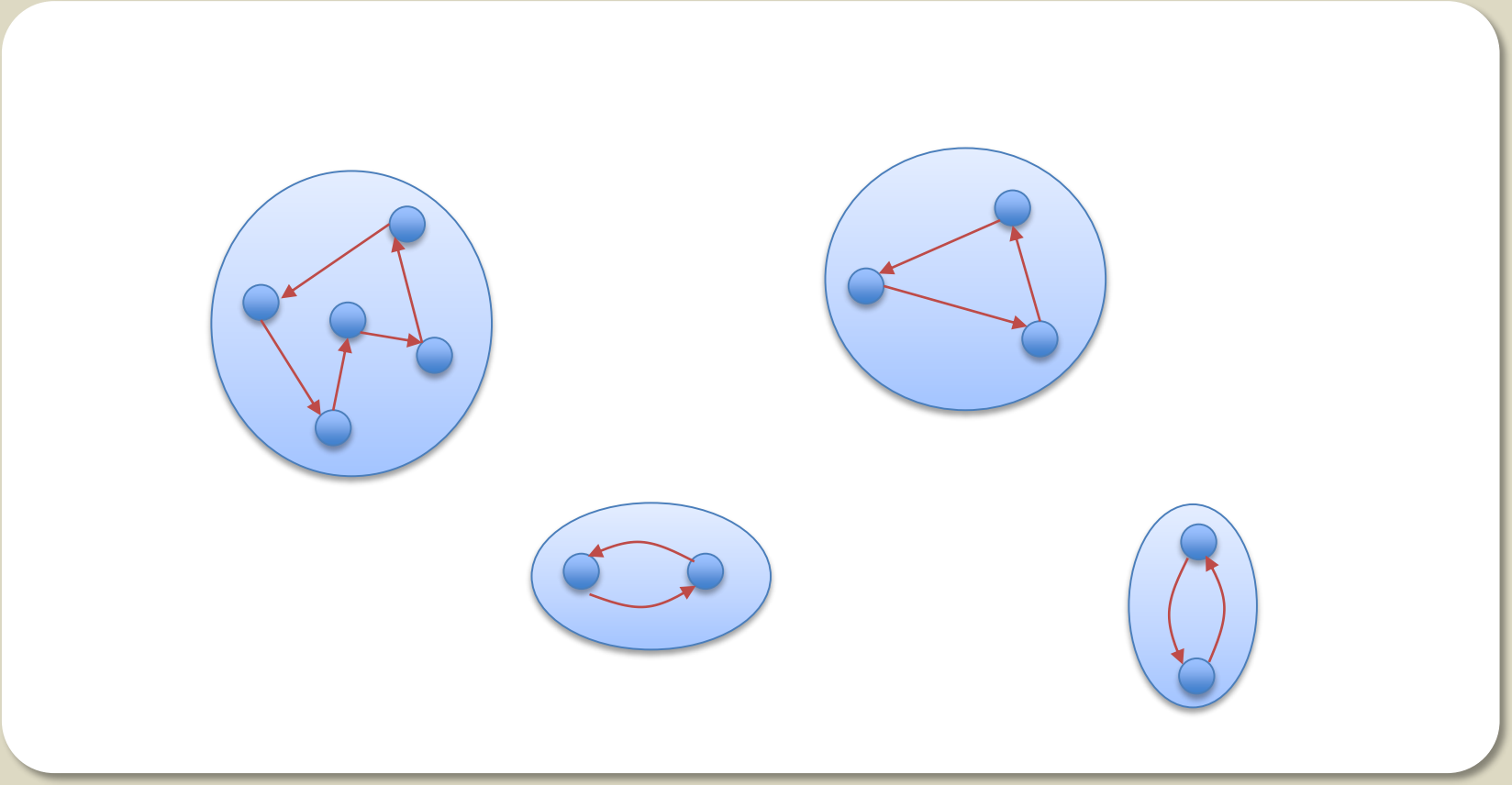

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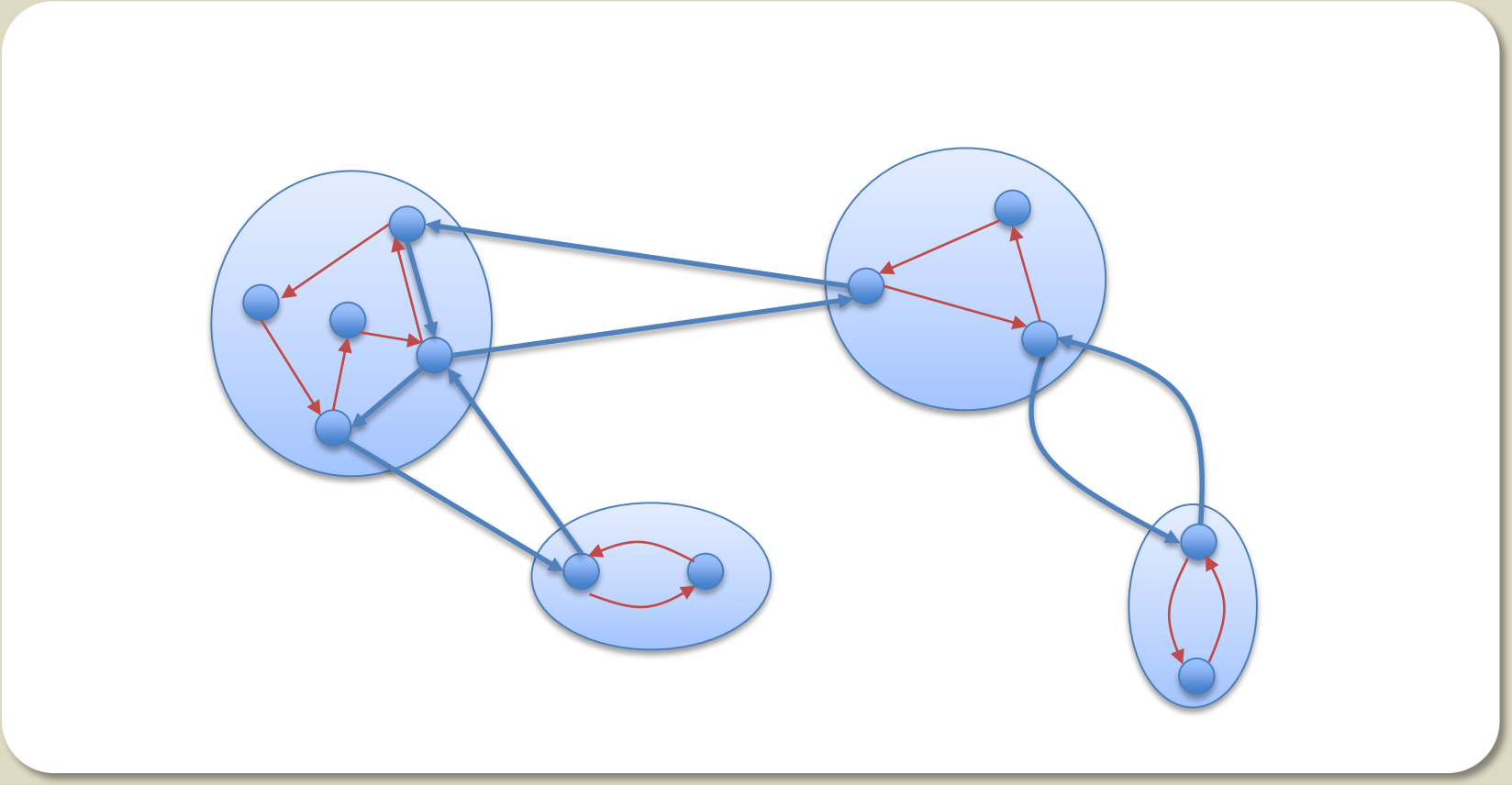

Repeatedly solve Local-Connectivity ATSP with the current connected subgraphs as partitions

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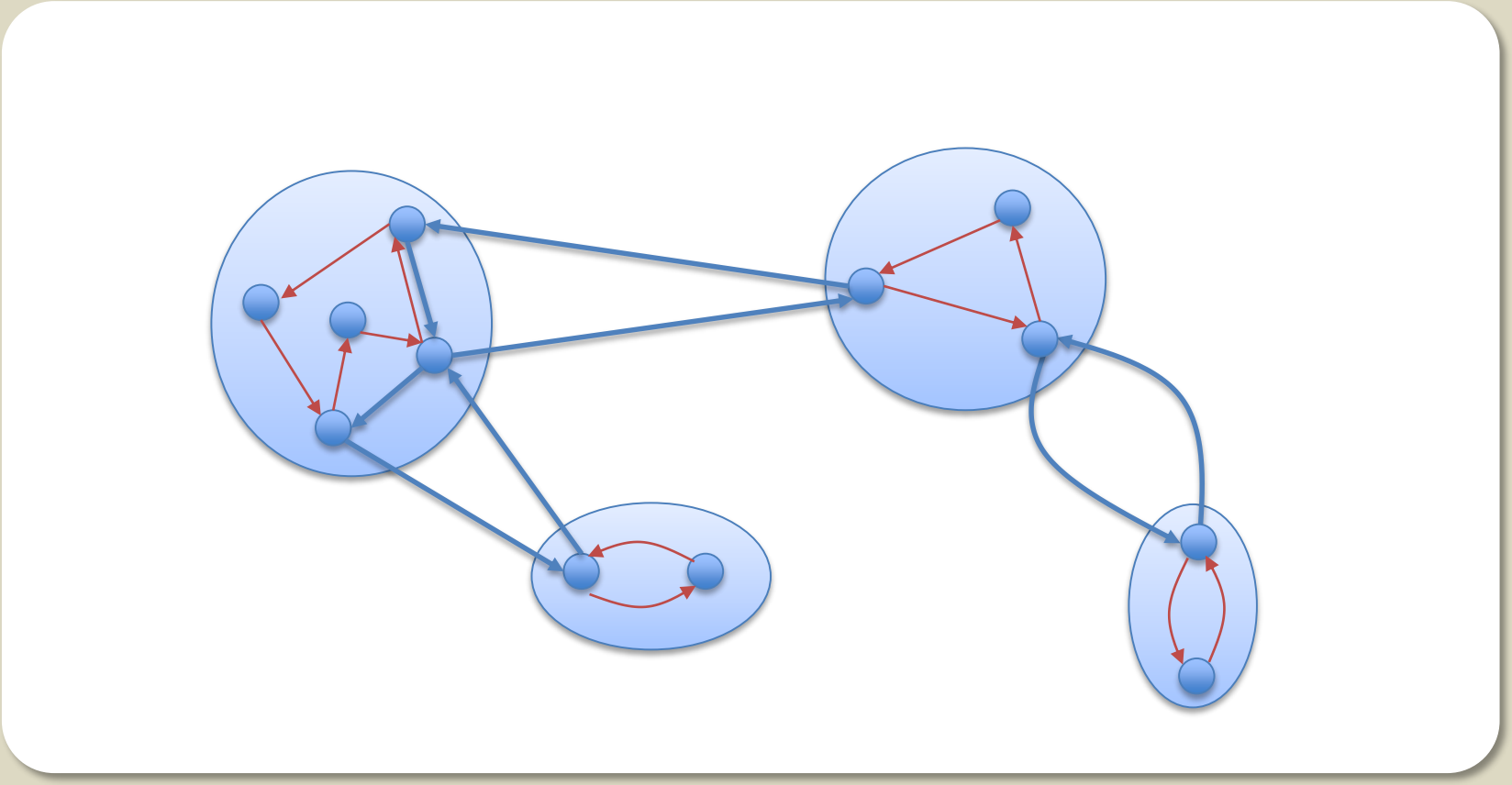
Cost of blue edges $\leq \alpha OPT$

Total cost $\leq \#iterations \cdot \alpha OPT \leq log(n) \cdot \alpha OPT$



Lexicographic Initialization

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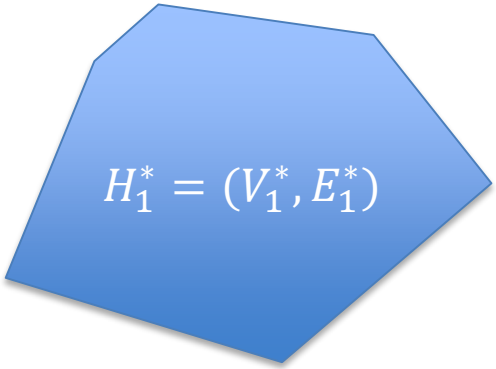
H_1^* = largest connected component that has 2α-light tour



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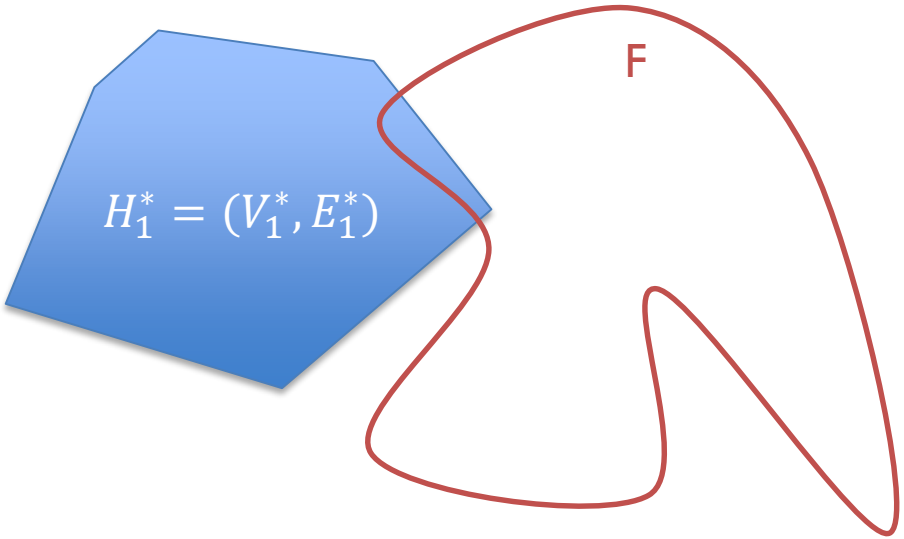

If red tour is α-light then $|\mathrm{F}| \leq \alpha |V_1^*|$

Why?

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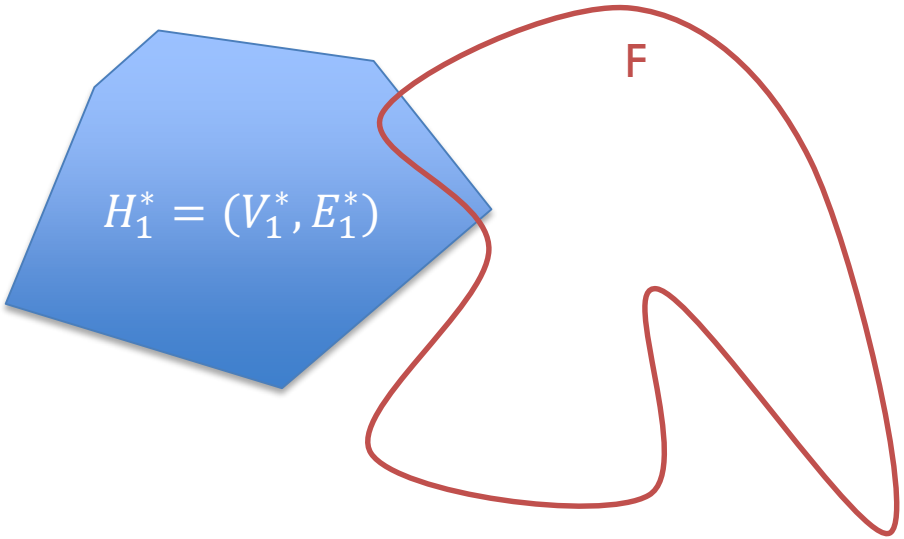


If red tour is α-light then $|\mathrm{F}| \leq \alpha |V_1^*|$

Why? Otherwise F spans more than $|V_1^*|$ vertices and this should be our largest component!

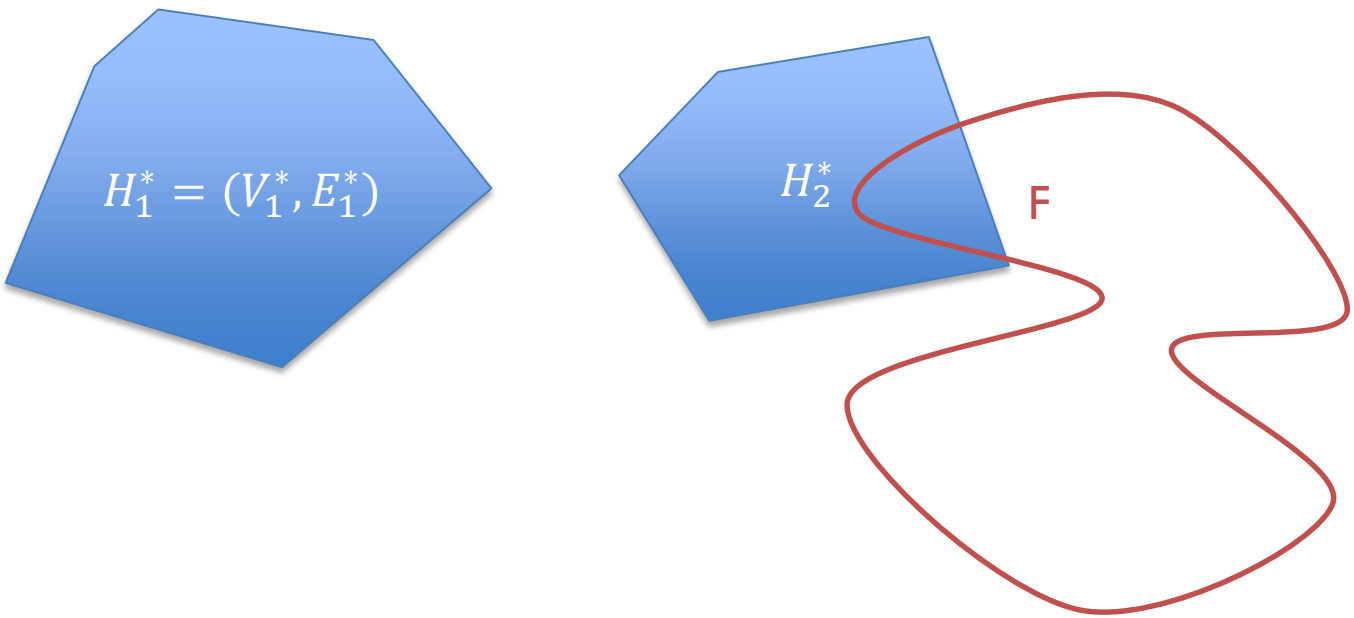
Lexicographic Initialization

H_1^* = **largest connected component that has 2α-light tour**

H_2^* = **largest disjoint connected component that has 2α-light tour**

Intersects H_2^* but not H_1^*:

$|F| \leq \alpha |V_2^*|$



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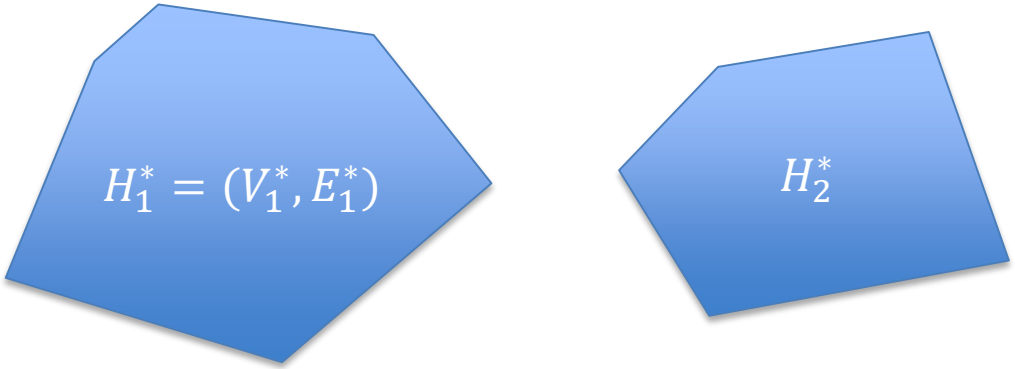

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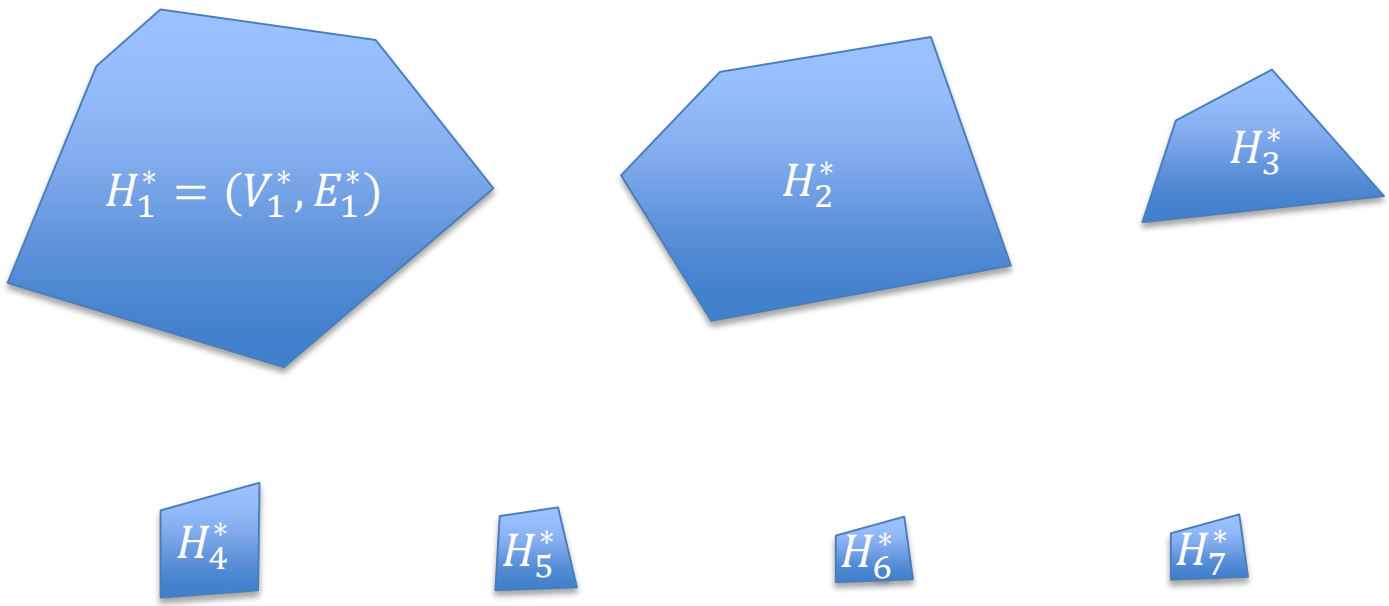
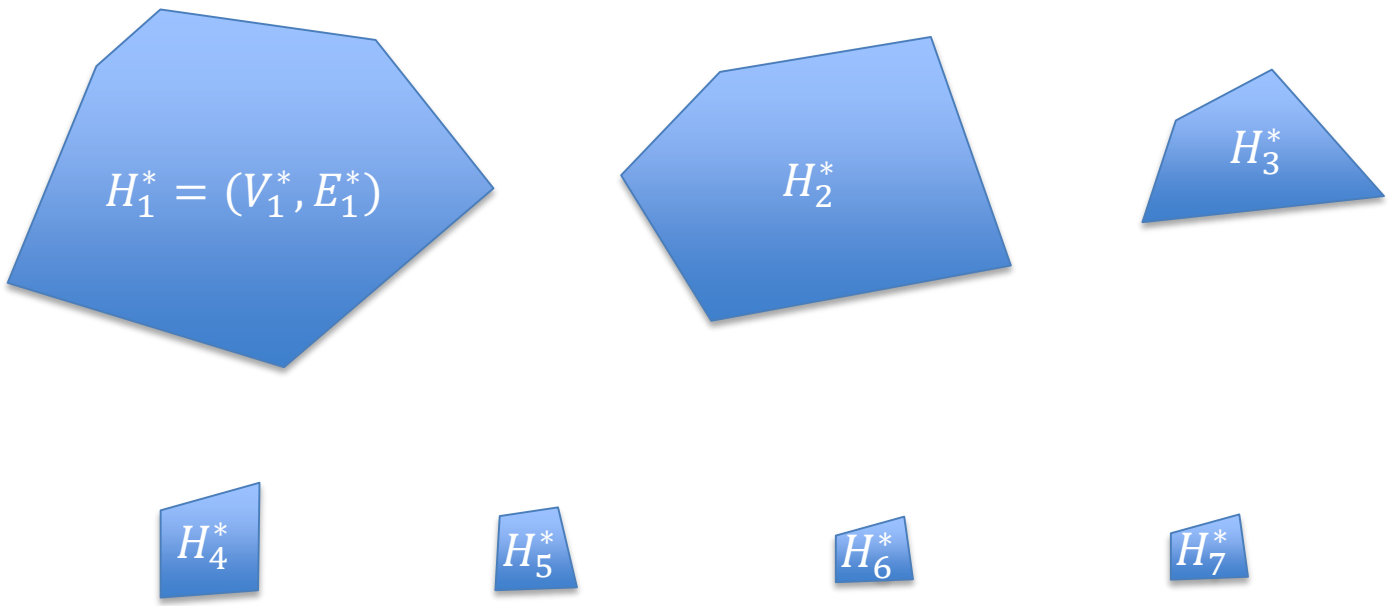
$\vdots$

$H_k^* =$ largest disjoint connected component that has 2α-light tour



Wish List for Merging Step

Cost of initialization at most $2\alpha OPT$ since each component is 2α-light



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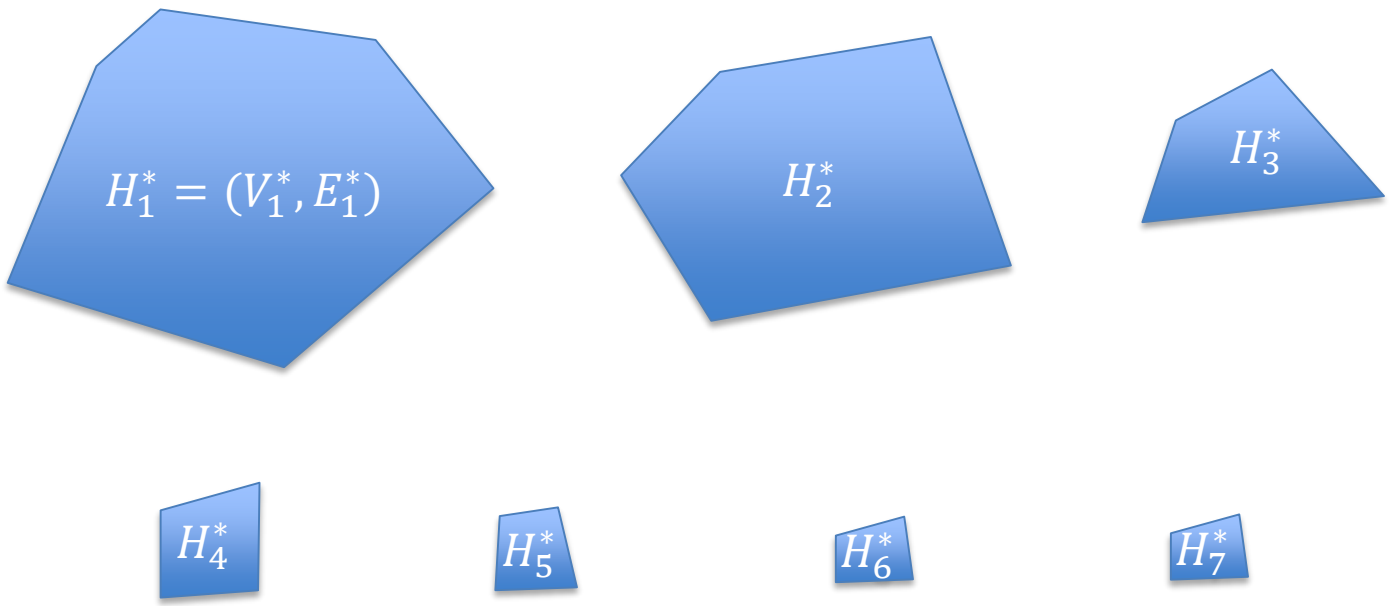
Suppose we can connect graph by adding α-light components so that we charge at most one to each component of initialization



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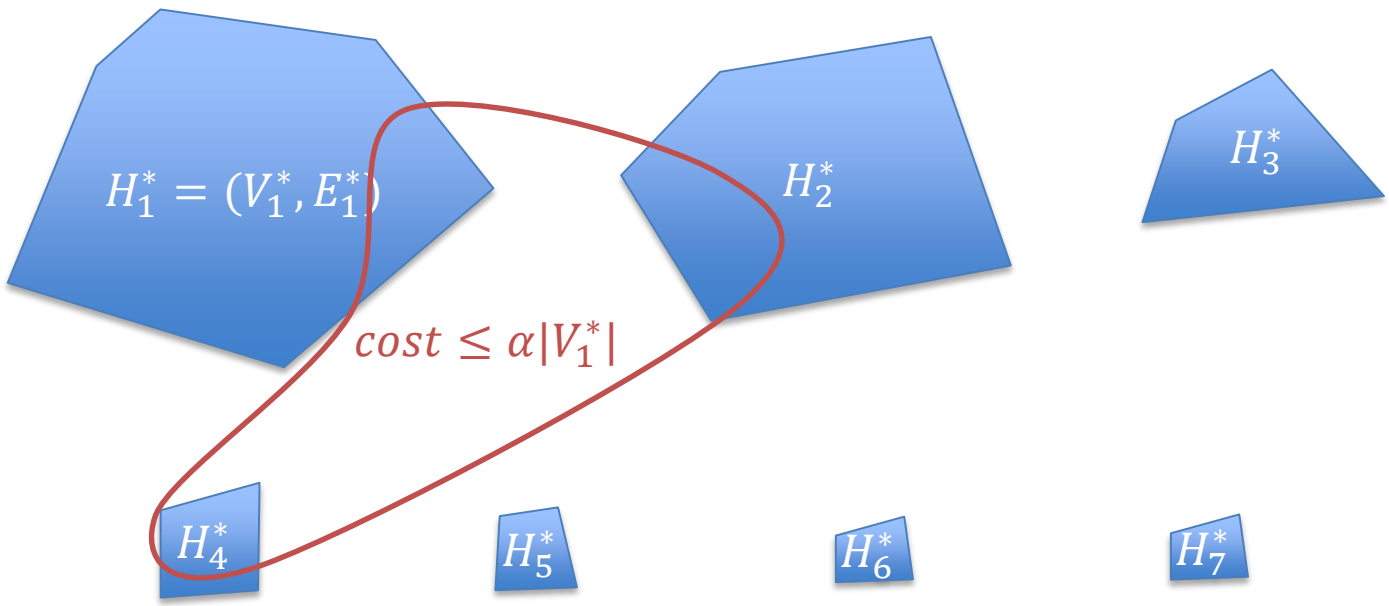

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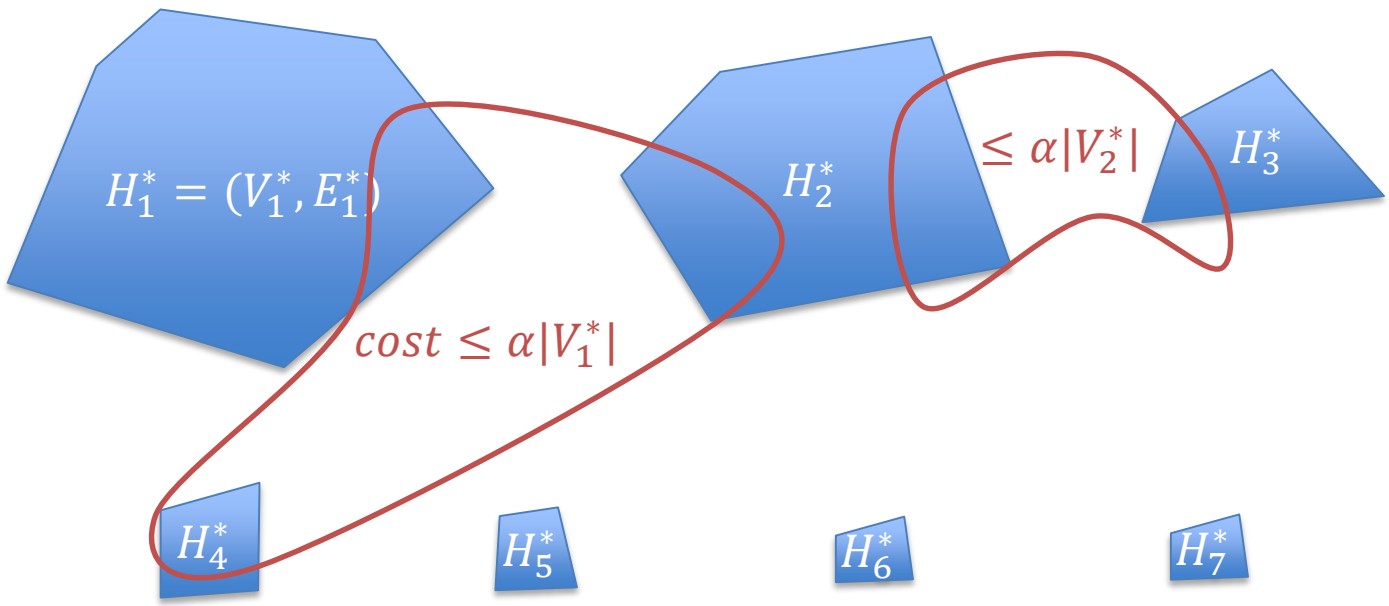

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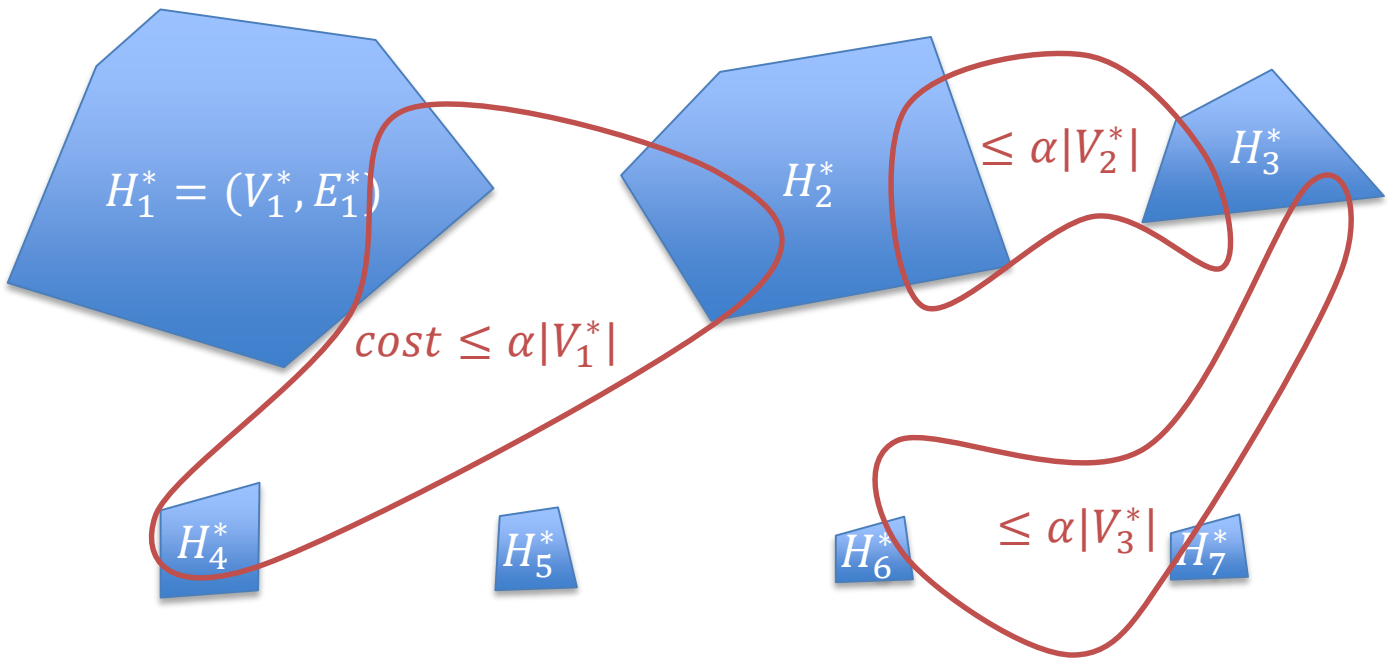

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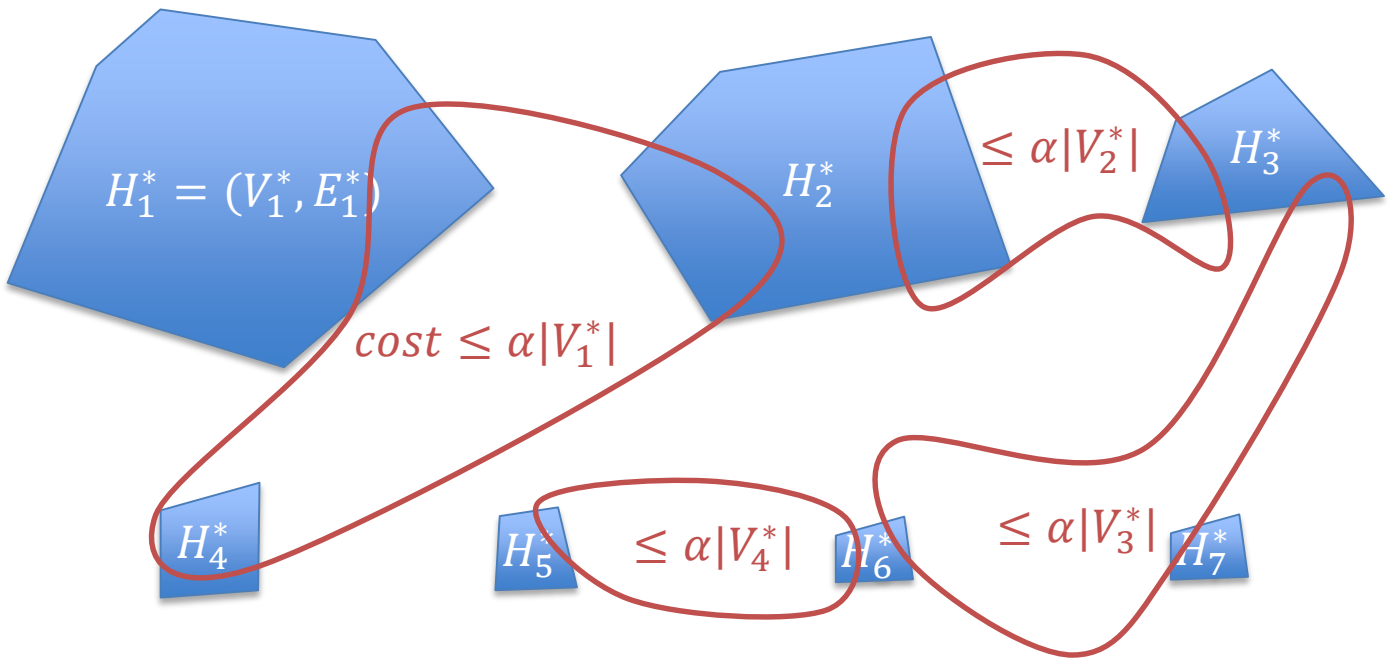

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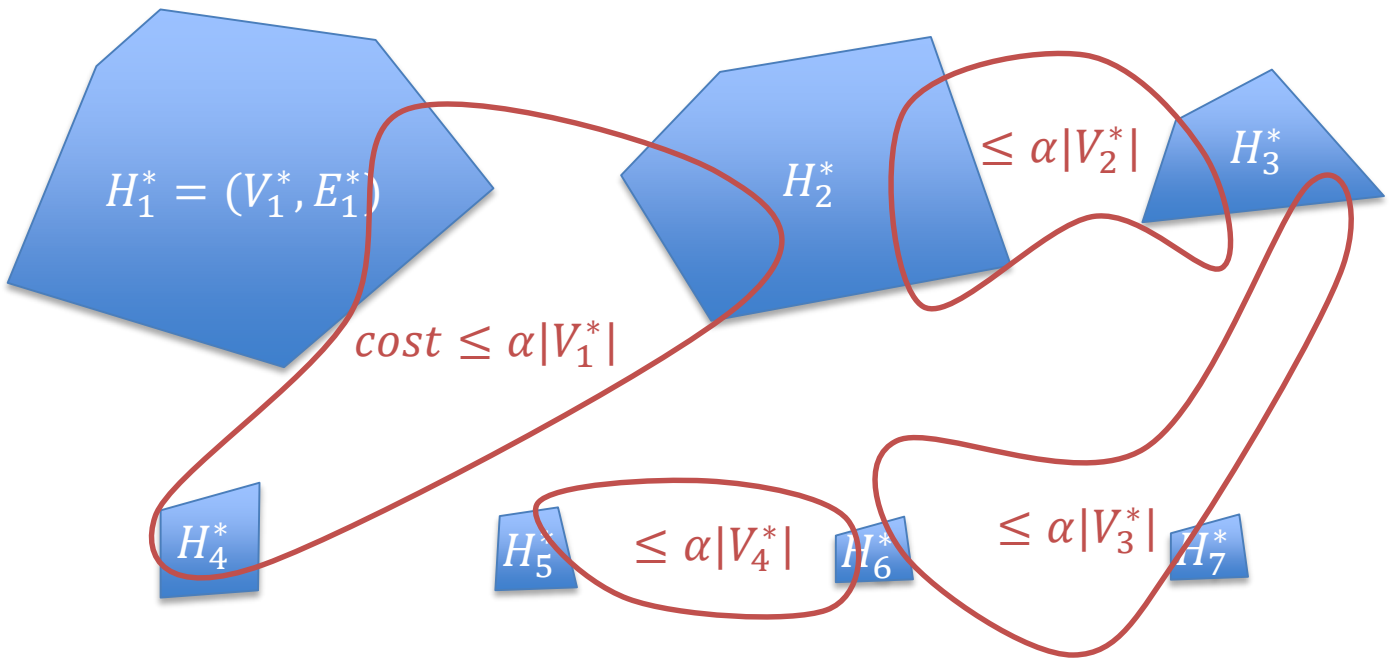
Suppose we can connect graph by adding α-light components so that we charge at most one to each component of initialization

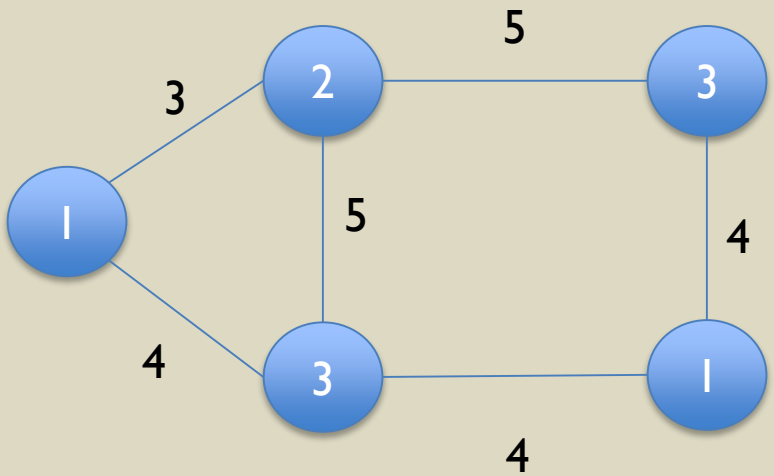
Then total cost $\leq 3\ \alpha OPT$



Some open problems…

Node-Weighted Symmetric TSP

- distance of $\{u, v\} \in E$ is $w(u) + w(v)$



Can you do better than Christofides (1.5)?

Constant for ATSP on General Metrics?

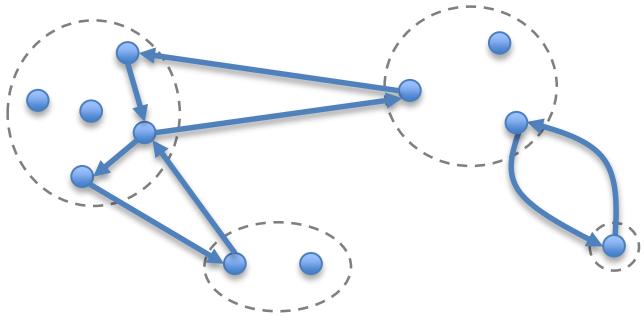
THIN TREE:

Does any k-edge connected graph have a $O(1)$-thin tree, i.e., a tree T such that for each $S \subset V$,

$$|\delta_T(S)| \leq \frac{\alpha}{k}|\delta_G(S)| \text{ ?}$$

LOCAL-CONNECTIVITY ATSP:

Can you design a $O(1)$-light algorithm for general metrics?



Summary

- Overview of Existing Approaches
- New approach that relaxes connectivity
- Integrality gap for node-weighted metrics is ≤ 13
- **Nice Open Questions** both for Symmetric and Asymmetric TSP