ADFOCS'20: Fair-division
Exercise 1: Competitive Equilibrium (CE)

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All the questions pertain to a Fisher instance given by:

- Set A of n agents.
- Set G of m goods, each with unit supply.
- Each agent $i \in A$ has B_i budget, and linear valuation function $v_i(x_i) = \sum_{j \in G} v_{ij} x_{ij}$.

In today's lecture we saw that prices $p = (p_1, \ldots, p_n)$ and allocation $X = (x_1, \ldots, x_n)$ constitute a competitive equilibrium iff

- *Optimal bundle:* $\forall i \in A, \;(p \cdot x_i) = B_i$, and,

$$\forall j \in G, \quad x_{ij} > 0 \Rightarrow \frac{v_{ij}}{p_j} = \max_{k \in G} \frac{v_{ik}}{p_k}$$

- *Demand = Supply:* $\forall j \in G, \;\; \sum_{i \in A} x_{ij} = 1$

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1. *(fairness properties)*

   Given a Fisher instance where budget of agent i is B_i (budget can also be thought of as weight/clout/importance for a fair-division task), show that a CE allocation is

   (a) weighted envy-free
   (b) weighted proportional
   (c) maximizes weighted Nash welfare

   Remark. For the last part it suffices to show that CE allocation gives a feasible point in the EG convex program (with weighted objective) and together with CE prices they satisfy the complementary slackness and dual-feasibility conditions.

2. *(Finite-time algorithm)*

   With respect to prices $p = (p_1, \ldots, p_n)$, let the MBB set be defined as $MBB(p) = \{(i, j) \in A \times G \mid \frac{v_ij}{p_j} = \max_{k \in G} \frac{v_{ik}}{p_k}\}$.

   (a) Given $MBB(p^*)$ for a competitive equilibrium price p^*, design an efficient algorithm to find p^* and corresponding equilibrium allocation X^*.
   (Hint: Linear feasibility program w/ dollar-spent variables (f_{ij}))

(b) Using solution to part (a) as a subroutine, design a finite time (exponential-time) algorithm to find a CE.

3. *(Proportional response (PR) dynamics)*

Consider a proportional response function $f : D \to D$, where $D = \{b \in \mathbb{R}_+^{mn} \mid \sum_{j \in G} b_{ij} = B_i\}$: for a $b \in D$, if $b' = f(b)$ then construct b' from b as follows:

Think of b_{ij} as the bid of agent i on good j. Price of good j is the total bids collected on it, i.e., $p_j = \sum_{i \in A} b_{ij}$. And the allocation is proportional to the bid, i.e., $x_{ij} = \frac{b_{ij}}{p_j}$. (In economics such a market implementation is known as *Trading-post*, introduced by Shapley and Shubik in 1977.)

Based on the utilities obtained, agents update their bids (proportional to the utility received from the previous bid):

$$b'_{ij} = B_i \frac{v_{ij} x_{ij}}{\sum_{k \in G} v_{ik} x_{ik}}$$

Show that, given a CE (p^*, X^*) the corresponding bids $b^*_{ij} = p^*_j x^*_{ij}$ for all (i,j) forms a fixed-point of f, i.e., $f(b^*) = b^*$.

Remark. In the proportional response dynamics agents update their bids as per the f function every day. That is, starting with arbitrary bid profile $b(0) \in D$, bids on day $t \geq 1$ is $b(t) = f(b(t-1))$. This is a well-studied dynamics that is known to converge to the fixed-point a.k.a. CE. The dynamics can be extended to more general utility functions like CES, gross-substitutes, and is known to converge for these too. See slides for the references.

4. *(Linear complementarity problem (LCP) formulation)*

An LCP (extension of linear program) is defined as follows: Given an $n \times n$ matrix M and n-dimensional vector q, find $y \in \mathbb{R}^n$ such that

$$(My)_i \leq q_i, \quad y_i \geq 0, \quad y_i(My - q)_i = 0, \quad \forall i \in \{1, \ldots, n\}$$

For short, we will write the above as

$$\forall i \in \{1, \ldots, n\}, \quad (My)_i \leq q_i \quad \perp \quad y_i \geq 0$$

Now consider the following LCP for the Fisher model, with variables corresponding to money spent (f_{ij}s), prices (p_js), and inverse MBB (λ_is)

$$\begin{array}{lll} \forall i \in A, & \sum_{j \in G} f_{ij} \geq B_i & \perp \quad \lambda_i \geq 0 \\ \forall j \in G, & \sum_{i \in A} f_{ij} \leq p_j & \perp \quad p_j \geq 0 \\ \forall (i,j) \in A \times G, & v_{ij}\lambda_i - p_j \leq 0 & \perp \quad f_{ij} \geq 0 \end{array}$$

Show that,

(a) If (p^*, X^*) is a CE then setting $p_j = p^*_j,\ \forall j, f_{ij} = x^*_{ij} p^*_j,\ \forall (i,j)$, and $\lambda_i = \min_k \frac{p^*_k}{v_{ik}},\ \forall i$ gives a solution of the LCP

(b) A solution of the LCP gives a CE.

5. *(Another convex program)*

Consider the following convex program in ($ spent) f_{ij} variables and (price) p_j variables.

$$\begin{array}{ll} max: & \sum_{i\in A, j\in G} f_{ij}\log(v_{ij}) - \sum_{j\in G} p_j \log p_j \\ s.t. & \sum_{i\in A} f_{ij} = p_j, \quad \forall j \in G \\ & \sum_{j\in G} f_{ij} = B_i, \quad \forall i \in A \\ & f_{ij} \geq 0, \quad \forall (i,j) \in A\times G \end{array}$$

Show that the solutions of the above formulation gives prices and ($ spent) allocation at a CE.

(Hint: Again use KKT)