ADFOCS'20: Fair-division
Exercise 2: Computation of Competitive Equilibrium

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1. Design an efficient algorithm to detect Event 2 in today's algorithm, namely *new MBB edge appears in the graph.*

2. Show that, for the case of Fisher model with binary valuations, *i.e.,* $v_{ij} \in \{0, 1\}$ for all $(i, j) \in A \times G$, the algorithm we discussed terminates in $O(n)$ many iterations of the outer while loop (recall that $n = |A|$); take the starting prices to be $p_j = \epsilon$, $\forall j \in G$ where $0 < \epsilon < \min_{i \in A} B_i/m$.

   Remark. Observe that both the events of our algorithm can be computed in *strongly polynomial-time.* Therefore, the proof of the above statement shows that the algorithm runs in strongly polynomial time for binary instances.

3. Consider a bi-valued HZ instance: for each agent i her value for good for good j is $v_{ij} \in \{a_i, b_i\}$ for all $j \in G$, where $0 \leq a_i < b_i$. Reduce the computation of HZ equilibrium for this instance to finding HZ equilibrium in a binary valued instance where for all (i, j) pairs $v_{ij} \in \{0, 1\}$.

4. (Recall) Spending restricted model is given as follows:

   - A: set of n agents
   - G: set of m divisible goods
   - For each agent $i \in A$, budget $B_i \geq 0$ and linear valuation function $v_i(x_{i1}, \dots, x_{im}) = \sum_{j \in G} v_{ij} x_{ij}$ where $v_{ij} > 0, \forall j \in G$. (Note that we have assumed all v_{ij}s to be strictly positive for simplicity.)
   - For each good $j \in G$, supply is one unit, and there is *spending-restriction* of $c_j > 0$ dollars. The c_js are such that, $\sum_{j \in G} c_j \geq \sum_{i \in A} B_i$.

   A competitive equilibrium is a (price, allocation) pair (p, X) where $p = (p_1, \dots, p_m)$ are the prices of the m goods and $X = (x_1, \dots, x_n)$ are the allocation vectors of the agents such that,

   - For each agent $i \in A$, x_i is her *optimal bundle* at prices p, i.e.,

     $$x_i \in argmax_{x:(p \cdot x) \leq B_i} v_i(x)$$

   - *Market clears, up to spending restriction*, i.e., for each good $j \in G$, $\sum_{i \in A} x_{ij} \leq 1$ and $\sum_{i \in A}(p_j x_{ij}) \leq c_j$, and

     $$\text{either} \quad \sum_{i \in A} x_{ij} = 1 \quad \text{or} \quad \sum_{i \in A}(p_j x_{ij}) = c_j$$

Modify the algorithm for the Fisher model that we discussed in the lecture to find a CE of a spending-restricted model.

(Hint: Think of introducing one more event (say Event 3) where you "freeze" the capacity on s to j edge for good j, while still increasing price p_j separately if needed to maintain MBB edges.)

5. *(Bonus problem.)* Recall the convex formulation to compute CE for Fisher model from Problem 5 of Exercise set 1. Following is a slight extension of this formulation.

$$
\begin{array}{ll}
max: & \sum_{i\in A, j\in G} f_{ij}\log(v_{ij}) - \sum_{j\in G}(q_j \log q_j - q_j) \\
s.t. & \quad \sum_{i\in A} f_{ij} = q_j, \quad \forall j \in G \\
 & \quad \sum_{j\in G} f_{ij} = B_i, \quad \forall i \in A \\
 & \quad q_j \le c_j, \quad \forall j \in G \\
 & \quad f_{ij} \ge 0, \quad \forall (i,j) \in A \times G
\end{array}
$$

Show that an optimal solution $((f_{ij})_{(i,j)\in A\times G}, (q_j)_{j\in G})$ of the above convex program captures a CE in terms of money allocations of agent i on good j in f_{ij} and total spending on good j in q_j at the equilibrium. The actual price of good j will come from an expression involving both q_j and the dual variable for constraint $q_j \le c_j$.

(Hint: Again use KKT)