Fair Division of Indivisible Items

Jugal Garg

UNIVERSITY OF ILLINOIS URBANA-CHAMPAIGN

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Maximin Share (MMS) [B11]

- Suppose we allow agent i to propose a partition of items into n bundles with the condition that i will choose at the end
- Clearly, i partitions items in a way that maximizes the value of her least preferred bundle
- μ_i := Maximum value of $i's$ least preferred bundle

- Π := Set of all partitions of items into n bundles
- $\mu_i := \max_{A \in \Pi} \min_{A_k \in A} v_i(A_k)$

- MMS Allocation: A is called MMS if $v_i(A_i) \geq \mu_i, \forall i$

What is Known?

- Finding MMS value is NP-hard
  - PTAS for finding MMS value [W97]

Existence (MMS allocation)?

- $n = 2$: YES EXERCISE
- $n > 2$: NO [PW14]

- α-MMS allocation: $v_i(A_i) \geq \alpha . \mu_i$
  - 2/3-MMS exists [PW14, AMNS17, BK17, KPW18, G.MT18]
  - 3/4-MMS exists [GHSSY18]
  - $(3/4 + 1/(12n))$-MMS exists [G.T20]

Properties

- Normalized valuations
  - Scale free: $v_{ij} \leftarrow c.v_{ij}, \forall j \in M$
  - $\sum_j v_{ij} = n \quad \Rightarrow \quad \mu_i \leq 1$
- Ordered Instance: We can assume that agents' order of preferences for items is same: $v_{i1} \geq v_{i2} \geq \cdots v_{im}, \forall i \in N$
- Valid Reduction (α-MMS): If there exists $S \subseteq M$ and $i^* \in N$
  - $v_{i^*}(S) \geq \alpha.\mu_{i^*}^n(M)$
  - $\mu_i^{n-1}(M \setminus S) \geq \mu_i^n(M), \forall i \neq i^*$

  $\Rightarrow$ We can reduce the instance size!

Challenge

- Allocation of high-value items!
- If for all $i \in N$
  - $v_i(M) = n \;\Rightarrow \mu_i \leq 1$
  - $v_{ij} \leq \epsilon, \forall i, j$

Bag Filling Algorithm for $(1-\epsilon)$-MMS allocation:

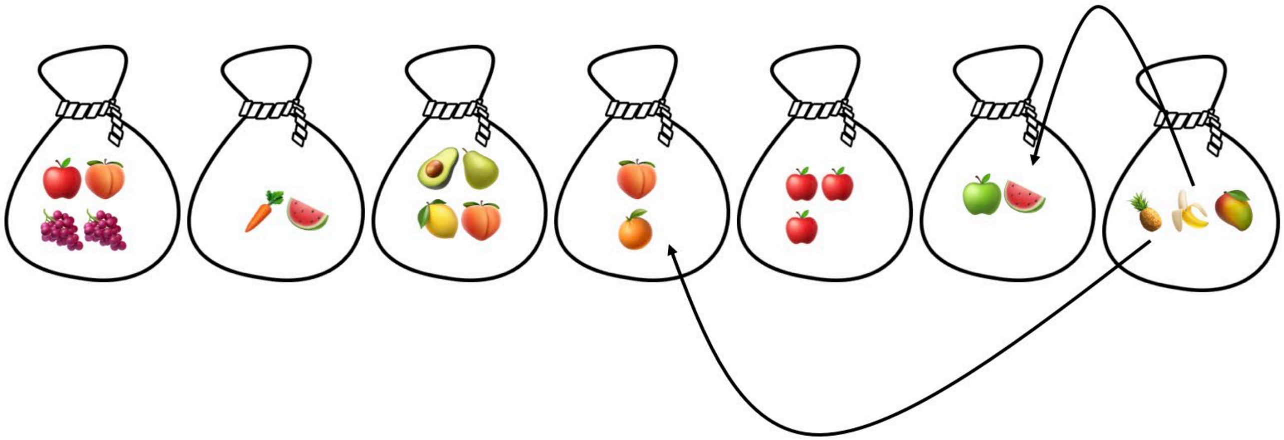
> Repeat until every agent is assigned a bag
> - Start with an empty bag B
> - Keep adding items to B until some agent i values it $\geq (1-\epsilon)$
> - Assign B to i and remove them

1/2-MMS Allocation

- Assume that μ_i is known for all i
  - Scale valuations so that $\mu_i = 1 \Rightarrow v_i(M) \geq n$

Step 1: Valid Reductions
- If $v_{i1} \geq 1/2$ then assign item 1 to i

Step 2: Bag Filling



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Step 2: Bag Filling

1/2-MMS Allocation

- μ_i is not known

Step 0: Normalize Valuations: $\sum_j v_{ij} = n \Rightarrow \mu_i \leq 1$

Step 1: Valid Reductions

- If $v_{i1} \geq 1/2$ then assign item 1 to i
- After every valid reduction, normalize valuations

Step 2: Bag Filling

2/3-MMS Allocation [G.MT19]

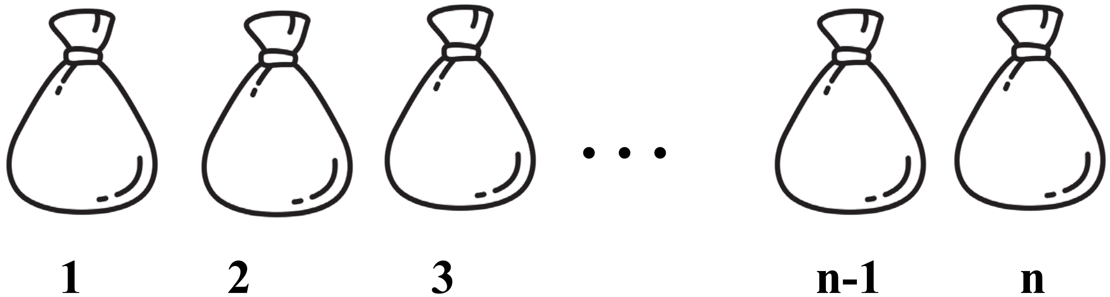
- Assume that μ_i is known for all i
  - Scale valuations so that $\mu_i = 1 \Rightarrow v_i(M) \geq n$
- If all $v_{ij} \leq 1/3$ then ?

Step 1: Valid Reductions

- If $v_{i1} \geq 2/3$ then assign item 1 to i
- If $v_{in} + v_{i(n+1)} \geq 2/3$ then assign $\{n, n+1\}$ to i

Step 2: Generalized Bag Filling

- Initialize n bags $\{B_1, \ldots B_n\}$ with $B_k = \{k\}, \forall k$



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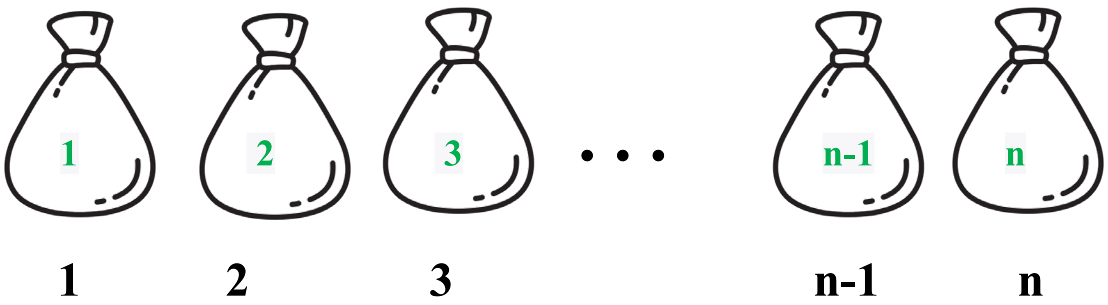

2/3-MMS Allocation [G.MT19]

- μ_i is not known

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Step 1: Valid Reductions

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New Fairness Notions

- n agents, m indivisible items (like cell phone, painting, etc.)
- Each agent i has a valuation function over subset of items denoted by $v_i : 2^m \to \mathbb{R}$
- Goal: fair and efficient allocation

Fairness:
Envy-free (EF)
Proportionality (Prop)

Efficiency:
Pareto optimal (PO)

Maximum Nash Welfare (MNW)

EF1 EFX	Lecture 3
MMS Prop1	Lecture 4
Guarantees	Lecture 5

Objectives

- Maximize the sum of valuations (**Utilitarian** Welfare): $$UW(A) = \sum_i v_i(A_i)$$

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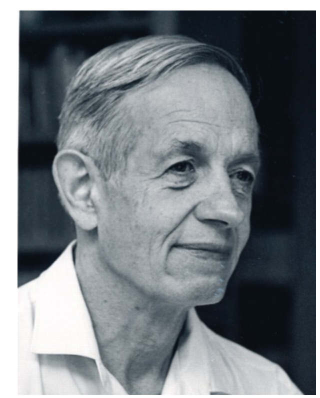
- Maximize the minimum of valuations
  (Max-Min-Fairness, **Egalitarian** Welfare):
  $$EW(A) = \min_i v_i(A_i)$$

- Maximize the geometric mean of valuations
  (≈ **Efficiency** + **Fairness**, **Maximum Nash Welfare**):
  $$\mathrm{N}W(A) = \left(\prod_{i \in A} v_i(A_i)\right)^{1/n}$$

Scale-free



Maximum Nash Welfare (MNW)

- Maximum Nash welfare (MNW): An allocation A that maximizes the Nash welfare among all feasible allocations i.e.,

$$A^* = \arg\max_A (\prod_i v_i(A_i))^{1/n}$$

Additive Valuations $(v_i(A_i) = \sum_{j \in A_i} v_{ij})$:

- Divisible Items: MNW ≡ CEEI ⇒ Envy-free + Prop + PO + …
- Indivisible Items: MNW ⇒ EF1 + PO + $\Omega(\frac{1}{\sqrt{n}})$-MMS [CKMPSW16]
  - Existence of EF1 + PO allocation

MNW (additive)

- APX-hard [Lee17]; 1.069-hardness [G.HM18]

Approximation:

- ρ-approximate MNW allocation A: $\rho.\mathrm{NW}(A) \geq MNW$
  - 2 [CG15, CDGJMVY17], e [AOSS17]
  - 1.45 [BKV18] (pEF1 approach)
- Fairness Guarantees
  - Prop1 + PO + $\frac{1}{2n}$-MMS + 2-MNW [G.M19]

Close the gaps!

MNW: Generalizations

- Non-symmetric Agents (different entitlements/weights)
  - Weighted envy-free, weighted proportionality
  - MNW (weighted geometric mean)

- Beyond Additive Valuations

Additive $\subset$ SC $\subset$ OXS $\subset$ Rado / Budget additive $\subset$ Submodular $\subset$ Subadditive

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Additive ⊂ SC ⊂ OXS ⊂ Rado ⊂ Submodular ⊂ Subadditive
Budget additive

The non-symmetric MNW Problem

- Non-symmetric MNW was proposed in [HS72, K77] and has been extensively studied and used in many applications
  - Agent i has a weight of w_i

$$\mathrm{NW}(A) = \left(\prod_i v_i(A_i)^{w_i} \right)^{1/\sum_i w_i}$$

weighted geometric mean of agents' valuations

- MNW $= \arg\max_A NW(A)$
- ρ-approximate MNW allocation A: $\rho.\, \mathrm{NW}(A) \geq \mathrm{MNW}$

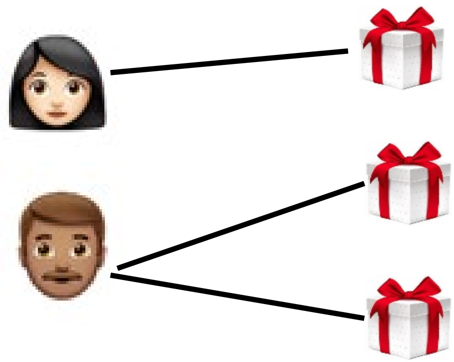
Example (additive)

w_i

A

1 [10, 10, 1]

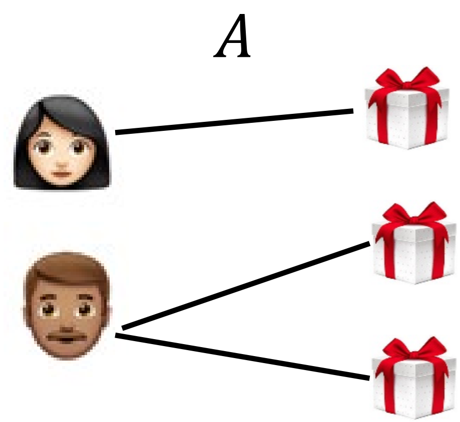
1 [1, 2, 1]



$$\text{MNW}=\text{NW}(A) = (10^1 \cdot 3^1)^{1/2}$$

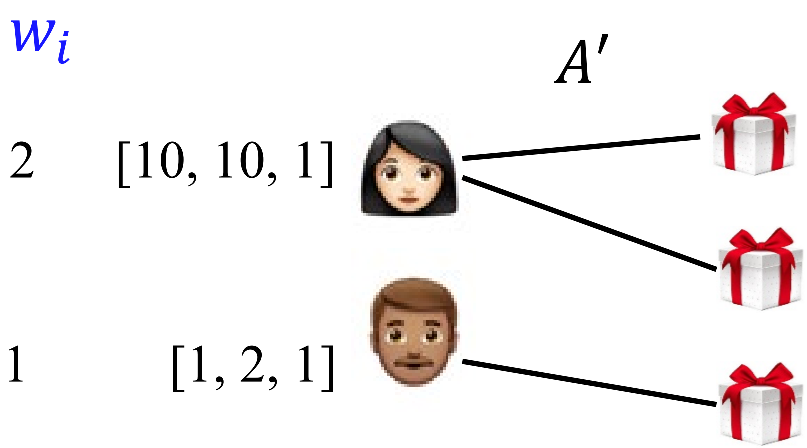
Example (additive)

w_i		
2	[10, 10, 1]	
1	[1, 2, 1]	



$$\text{NW}(A) = (10^2 \cdot 3^1)^{1/3}$$

Example (additive)

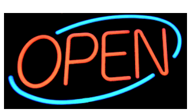


$$\text{NW}(A) = (10^2 \cdot 3^1)^{1/3} < (20^2 \cdot 1^1)^{1/3} = \text{NW}(A') = \text{MNW}$$

MNW Approximations: Additive

	Lower bound	Upper Bound
Symmetric	1.069	1.45
Non-symmetric	1.069	$O(n)$

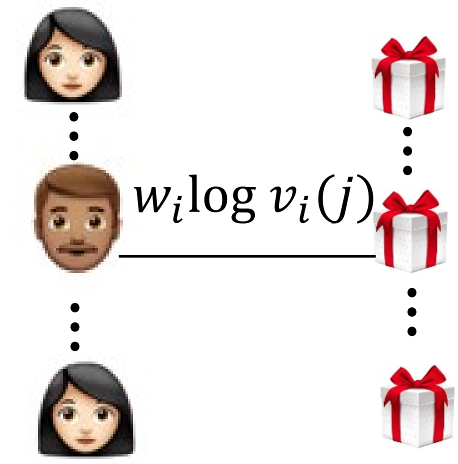
n: # of agents



Constant factor? sublinear?

Matching ($m = n$)

$$\text{NW}(A) = \left(\prod_i v_i(A_i)^{w_i}\right)^{1/\sum_i w_i}$$

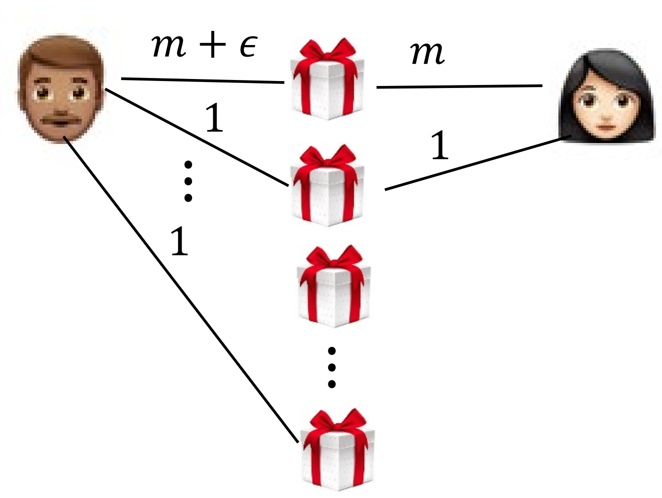


$$\text{MNW} = \max_A \ \text{NW}(A) \ \equiv \ \max_A \sum_i w_i \log v_i(A_i)$$

Claim: If $m = n$, then max-weight matching outputs MNW

$m > n$

- How good is max-weight matching?



$\text{NW}(A^*) \simeq m$

$\text{NW}(A) \simeq \sqrt{2m}$

- Issue: Allocation of high-value items!

Round Robin Procedure

- H_i: Set of n highest-valued items for agent i
- $u_i = v_i(M \setminus H_i)$
- Guarantee?

- H_i: Set of n highest-valued items for agent i
- $u_i = v_i(M \setminus H_i)$
- Round-Robin guarantees $\geq \frac{u_i}{n}$

MNW allocation A^*:

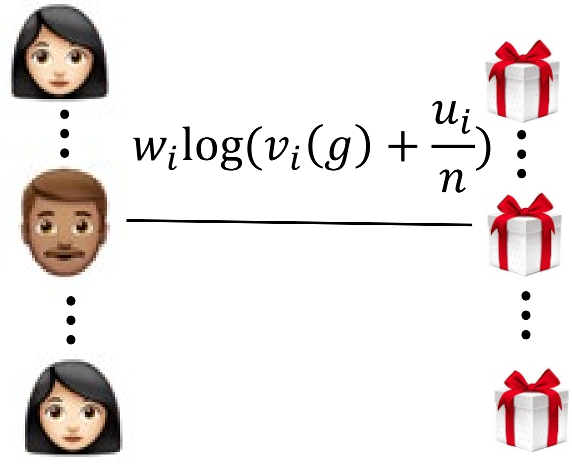
- g_i^*: highest-valued item in A_i^*
- $v_i(A_i^*) \leq n v_i(g_i^*) + u_i$

$$\leq n\left(v_i(g_i^*) + \frac{u_i}{n}\right)$$

- If $v_i(A_i) \geq v_i(g_i^*) + \frac{u_i}{n}$, then A is $O(n)$-approximation!

Matching + Round-Robin [G.KK20]

- H_i: Set of $2n$ highest-valued items for agent i
- $u_i = v_i(M \setminus H_i)$
- Allocate one item to each agent using max-weight matching with weights $w_i \log(v_i(g) + \frac{u_i}{n})$: y_i^* is allocated to i
- $A \leftarrow$ Allocate remaining items using Round Robin



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- $A \leftarrow$ Allocate remaining items using Round Robin

- g_i^*: highest-valued item in A_i^*
- $v_i(A_i^*) \leq 2n v_i(g_i^*) + u_i \implies \text{MNW} \leq 2n \left(\prod_i \left(v_i(g_i^*) + \frac{u_i}{n}\right)^{w_i}\right)^{1/\sum_i w_i}$
- $v_i(A_i) \geq v_i(y_i^*) + \frac{u_i}{n}$

$$\implies \text{NW}(A) \geq \left(\prod_i \left(v_i(y_i^*) + \frac{u_i}{n}\right)^{w_i}\right)^{\frac{1}{\sum_i w_i}} \geq \left(\prod_i \left(v_i(g_i^*) + \frac{u_i}{n}\right)^{w_i}\right)^{\frac{1}{\sum_i w_i}}$$

Theorem [G.KK20]: A is $2n$-MNW + EF1

Generalizations

- Non-symmetric Agents (different entitlements/weights)
  - Weighted envy-free, weighted proportionality
  - MNW (weighted geometric mean)

- Beyond Additive

Additive ⊂ SC ⊂ OXS ⊂ Rado / Budget additive ⊂ Submodular ⊂ Subadditive

non-negative monotone: $v(S) \leq v(T), \quad S \subseteq T$

Subadditive: $v(A \cup B) \leq v(A) + v(B), \quad \forall A, B$

Additive valuations are restrictive

100

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100

100

Additive valuations are restrictive

100

+

100

$125 \neq 100 + 100$

MNW Approximations: Symmetric Agents

Additive ⊂ SC ⊂ OXS ⊂ Rado / Budget additive ⊂ Submodular ⊂ Subadditive

Valuation	Lower bound	Upper Bound
Additive Budget additive Separable concave	1.069	1.45
OXS Rado	1.069	$O(1)$
Submodular	1.58	$O(n)$
Subadditive	$O(n^{1-\epsilon})$	$O(n)$

n: # of agents

MNW Approximations: Non-symmetric Agents

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Envy-free (EF) Allocation

Claim: An EF allocation A is $O(n)$-approximation

½-EFX Allocation

- ½-EFX allocation A: $v_i(A_i) \geq \frac{1}{2} v_i\left(A_j \setminus g\right), \forall g \in A_j, \forall i, j$

Claim: If $|A_i| \geq 2, \forall i,$ then A is $O(n)$-approximation

$O(n)$ Algorithm [CG.M20]

- H_i : Set of n highest-valued items for agent i
- Allocate one item per agent using max-weight matching with weights $w_i \log(v_i(g) + \frac{v_i(M \backslash H_i)}{n})$: y_i^* is allocated to i
- $A \leftarrow$ Allocate remaining items using ½-EFX algorithm

Claim: A is $O(n)$-MNW and ½-EFX allocation

Claim: A is $O(n)$-MNW

Proof (sketch):

- $Y \leftarrow \cup_i y_i^*$; g_i^*: highest-valued item in MNW allocation A_i^*
- $v_i(A_i^*) \leq \ nv_i(g_i^*) + v_i(M \setminus H_i) = n\left(v_i(g_i^*) + \frac{v_i(M\setminus H_i)}{n}\right)$

$$\Longrightarrow \ \text{MNW} \leq \ n\left(\prod_i \left(v_i(g_i^*) + \frac{v_i(M\setminus H_i)}{n}\right)^{w_i}\right)^{1/\sum_i w_i}$$

- $v_i(A_i) \geq v_i(y_i^*)$
- $v_i(A_i) \geq \frac{v_i(M\setminus Y)}{4n} \geq \frac{v_i(M\setminus H_i) - nv_i(y_i^*)}{4n}$

- $v_i(A_i) \geq \frac{1}{8}\left(v_i(y_i^*) + \frac{v_i(M\setminus H_i)}{n}\right)$

$$\text{NW}(A) \geq \frac{1}{8}\left(\prod_i \left(v_i(y_i^*) + \frac{v_i(M\setminus H_i)}{n}\right)^{w_i}\ \right)^{\frac{1}{\sum_i w_i}} \geq \frac{1}{8}\left(\prod_i \left(v_i(g_i^*) + \frac{v_i(M\setminus H_i)}{n}\right)^{w_i}\ \right)^{\frac{1}{\sum_i w_i}}$$

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MNW Approximations: Non-symmetric Agents

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