Box-Simplex Games: Algorithms, Applications, and Algorithmic Graph Theory Exercises #1

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Problem #1: $\|\cdot\|_{\text{op},\infty}$?

- Part (a): prove that $\|\mathbf{A}\|_{\text{op},\infty}$ is the maximum ℓ_1-norm of a row of $\mathbf{A}$.
- Part (b): for all $\mathbf{A} \in \mathbb{R}^{m\times n}$ let $\|\mathbf{A}\|_{\text{op},1} \stackrel{\text{def}}{=} \max_{x\in\mathbb{R}^n | x\neq 0} \frac{\|\mathbf{A}x\|_1}{\|x\|_1}$. Prove that $\|\mathbf{A}\|_{\text{op},\infty} = \|\mathbf{A}^\top\|_{\text{op},1}$.
- Part (c): provide a method which given $b \in \mathbb{R}^n$ and $c \in \mathbb{R}^m$ compute points $(x, y) \in B_\infty^n \times \Delta^m$ that are an $2\|\mathbf{A}\|_{\text{op},\infty}$-approximate saddle point for any $\mathbf{A} \in \mathbb{R}^{m\times n}$.

Problem #2: ℓ_1-Regression

Provide an algorithm which given any $\mathbf{A} \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^n$, and $\epsilon > 0$ computes an ϵ-additive approximation to the ℓ_1-constrained ℓ_1-minimization problem $\min_{x\in\mathbb{R}^n \,|\, \|x\|_1\leq 1} \|\mathbf{A}x - b\|_1$ in time $\tilde{O}(\text{nnz}(\mathbf{A})\|\mathbf{A}\|_1/\epsilon)$.

Hint: see previous problem.

Note: this technique has been used for solving minimum cost transshipment.[2].

Problem #3: Optimal Transport

In the *optimal transport problem* we are given $p, q \in \Delta^n$ and cost matrix $\mathbf{C} \in \mathbb{R}^{n\times n}$ with $|[\mathbf{C}]_{ij}| \leq C_{\max}$ for all $i, j \in [n]$. The goal is to compute a mapping from p to q of minimum cost. Formally, we call $\mathbf{X}$ a *valid transport map* if $\mathbf{X}\vec{1} = p$ and $\mathbf{X}^\top\vec{1} = q$ ($\mathbf{X}_{ij}$ specifies how much of p_i is mapped to q_j). We call $\mathbf{X} \bullet \mathbf{C} = \sum_{i,j\in[n]}[\mathbf{X}_{ij}] \cdot [\mathbf{C}]_{ij}$ the *cost* of the transport map and let OPT denote the minimum cost achievable by a transport map. Provide an algorithm which computes a transport map of cost $\text{OPT} + \epsilon \cdot C_{\max}$ in time $\tilde{O}(n^2\epsilon^{-1})$.

Hint: it may be helpful to view this as a type of perfect matching problem with general demands.

Note: the optimal transport problem is well-studied and has been the subject of extensive research. The approach taken here is closely related to [4, 2].

Problem #4: Width-dependent Packing LPs

For $\mathbf{A} \in \mathbb{R}_{\geq 0}^{m \times n}$ we call the following linear program a *packing linear program*

$$\mathrm{OPT}_{\mathrm{pack}} = \max_{x \in \mathbb{R}^n_{\geq 0} \,|\, \mathbf{A}x \geq \vec{1}} \vec{1}^\top x \,.$$

We call $x_\epsilon \in \mathbb{R}^n_{\geq 0}$ a $(1-\epsilon)$-*multiplicative approximate maximizer* if $\mathbf{A}x_\epsilon \geq \vec{1}$ and $c^\top \vec{1} \geq (1-\epsilon)\mathrm{OPT}_{\mathrm{pack}}$. Provide an algorithm which computes a $(1-\epsilon)$-approximate maximizer in time $\tilde{O}(\mathrm{nnz}(\mathbf{A})\mathrm{OPT}_{\mathrm{pack}}\|\mathbf{A}^\top\|_\infty \epsilon^{-1})$.

Hint: It may be helpful to solve multiple box-simplex problems. For a slightly easier problem, you may suppose that $\mathrm{OPT}_{\mathrm{pack}}$ *is known exactly.*

Note: More broadly a packing linear program is typically defined as $\max_{x \in \mathbb{R}^n_{\geq 0} \,|\, \mathbf{A}x \geq b} c^\top x$ *for* $\mathbf{A} \in \mathbb{R}^n_{\geq 0}$*,* $b \in \mathbb{R}^n_{\geq 0}$*, and* $c \in \mathbb{R}^m_{\geq 0}$*. This is reducible to the setting we consider by scaling and handling* 0 *entries, though this reduction it may change* $\|\mathbf{A}\|_{\mathrm{op},\infty}$*. Also, interestingly there are known, improved algorithms for solving this problem in* $\tilde{O}(\mathrm{nnz}(\mathbf{A})\epsilon^{-1})$ *[1] and broader generalizations to packing-covering linear programs [3].*

References

[1] Zeyuan Allen-Zhu and Lorenzo Orecchia. Nearly linear-time packing and covering LP solvers - achieving width-independence and -convergence. *Math. Program.*, 175(1-2):307–353, 2019.

[2] Sepehr Assadi, Yujia Jin, Aaron Sidford, and Kevin Tian. Semi-streaming bipartite matching in fewer passes and optimal space. *arXiv preprint arXiv:2011.03495*, 2021.

[3] Digvijay Boob, Saurabh Sawlani, and Di Wang. Faster width-dependent algorithm for mixed packing and covering lps. In Hanna M. Wallach, Hugo Larochelle, Alina Beygelzimer, Florence d'Alché-Buc, Emily B. Fox, and Roman Garnett, editors, *Advances in Neural Information Processing Systems 32: Annual Conference on Neural Information Processing Systems 2019, NeurIPS 2019, December 8-14, 2019, Vancouver, BC, Canada*, pages 15253–15262, 2019.

[4] Arun Jambulapati, Aaron Sidford, and Kevin Tian. A direct tilde{O}(1/epsilon) iteration parallel algorithm for optimal transport. In Hanna M. Wallach, Hugo Larochelle, Alina Beygelzimer, Florence d'Alché-Buc, Emily B. Fox, and Roman Garnett, editors, *Advances in Neural Information Processing Systems 32: Annual Conference on Neural Information Processing Systems 2019, NeurIPS 2019, December 8-14, 2019, Vancouver, BC, Canada*, pages 11355–11366, 2019.