Chapter 2, Cont.

OMv Conjecture implies OuMv Conjecture

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Notice:
Technique similar to APSP
→ Negative Triangle
RECAP FROM LAST TIME
OMv Conjecture

(Online Matrix-Vector Multiplication) [Henzinger, Krinninger, N, Saranurak, STOC’15]

Input: $n \times n$ Boolean matrix $M$
Then: $n$ Boolean vectors $v_i$

Output: $Mv_1$ $Mv_2$ $Mv_n$

Conjecture: No algorithms with total time $O(n^{3-\epsilon})$

Current Best: $O(n^3 / 2^{\sqrt{\log n}})$ [Larson-Williams SODA’17]
OuMv Conjecture (Matrix Form)

**Input:** \( n \times n \) Boolean matrix \( M \)

**Then:** \( n \) pairs of Boolean vectors \( (u_i, v_i) \)

**Output:** \( u_i^T M v_i \)

**Conjecture:** No algorithms with total time \( O(n^{3-\epsilon}) \) even with polynomial time to process \( M \)!
Theorem 1: O\(Mv\) conjecture holds with polynomial preprocessing time.

**Details:**

- **OMv Conjecture (recall):** No algorithms with total time \(O(n^{3-\epsilon})\).

- **OMv’ Conjecture:** No algorithms with total time \(O(n^{3-\epsilon})\), *even with polynomial time to process M!*

**Claim:** O\(Mv\) Conjecture implies O\(Mv’\) Conjecture

**Idea:** Partition M!
Theorem 1: OMv conjecture holds with polynomial preprocessing time.

Main Idea: Divide M into submatrices of small dimension $n'$
- Preprocessing time $\text{poly}(n') \ll n^2$ per element in M (when $n'$ small enough)
- Time for each multiplication is still better than the trivial $(n')^3$ time if $n'$ is not too small.

Proof (sketched):
1. Suppose algorithm $A$ can solve OMv with $n^{10}$ time for preprocessing M and $O(n^{2.9})$ time after that. We will construct an algorithm with $O(n^{2.99})$ time in total.
2. Divide M into submatrices each of dimension $n' \times n'$, where $n' = n^{1/20}$.

For each submatrix $M'$:
   a. Preprocess in $(n')^{10} = n^{1/2}$ time.
      - There are at most $n^2$ submatrices $\Rightarrow O(n^{2.5})$ total preprocessing time
   b. Handle every group of $n'$ vectors $v_i$ in $(n')^{2.9}$ time.
      - Details: “Roll-back” to the preprocessing stage after every $n'$ pairs.
      - Time over $n/n'$ groups and $\left(\frac{n}{n'}\right)^2 : (n')^{2.9} \times \left(\frac{n}{n'}\right)^3 = n^{3-1/200}$
Claim 2: OMv’ conjecture implies OuMv conjecture
OMv’ conjecture implies OuMv conjecture

Idea: Suppose we can compute each $uMv$ in $n^{1.9}$ time. For each vector $v$, use $u$ to binary search for $i$ s.t. $(Mv)_i = 1$.

- Problem: To find all 1’s in $Mv$ needs to compute $uMv$ up to $n$ times $\rightarrow n^{2.9}$ to compute one $Mv$.
- Fix: Partition $M$!
OMv’ conjecture implies OuMv conjecture

Algorithm Sketch:
1. Divide M into submatrices each of dimension $n' \times n'$, where $n' = n^{1/2}$.
2. To find $i \leq \sqrt{n}$ s.t. $(Mv)_i = 1$, compute $u'M'v'$ with $u' = (1, 1, \ldots)$ on each submatrix $M'$ on top $\sqrt{n}$ rows.
3. $u'M'v' = 0 \rightarrow$ Ignore $M'$ from now on.
4. $u'M'v' = 1 \rightarrow$ Continue binary search within $M'$.
5. Repeat with coordinate $i$ ignored. Do the same for other rows.
Analysis

Algorithm Sketch:
1. Divide M into submatrices each of dimension $n' \times n'$, where $n' = n^{1/2}$.
2. To find $i \leq \sqrt{n}$ s.t. $(Mv)_i = 1$, compute $u'M'v'$ with $u' = (1, 1, ...)$ on each submatrix $M'$ on top $\sqrt{n}$ rows.
3. $u'M'v' = 0 \rightarrow$ Ignore $M'$ from now on
4. $u'M'v' = 1 \rightarrow$ Continue binary search within $M'$
5. Repeat with coordinate $i$ ignored. Do the same for other rows.

**Time to compute each $Mv$:** Suppose $u'M'v'$ takes $(n')^{1.9}$. For each matrix $M'$:
1. $u'M'v' = 0 \rightarrow$ Ignore $M'$ from now on $\rightarrow$ Happens only once for each $M'$, total time $= (n')^{1.9} \times \left(\frac{n}{n'}\right)^2 \ll n^2$
2. $u'M'v' = 1 \rightarrow$ Continue binary search within $M'$ $\rightarrow$ Will find new coordinate, thus happens only once for each coordinate. Total time $= (n')^{1.9} \times n \ll n^2$. 
Questions?

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