Exercises for ADFOCS 2018 - Sheet 2

Exercise 1 Metricity Problem: Given an $n \times n$ matrix $A$ with entries in $\{0, \ldots, n^c\}$ (for some large constant $c > 0$), decide whether for all $i, j, k \in [n]$ we have $A_{ij} \leq A_{ik} + A_{kj}$.

Prove that Metricity is equivalent to APSP under subcubic reductions.

Exercise 2 $X + Y$ problem: Given sets $X$ and $Y$ consisting of $n$ integers, decide whether the set $X + Y = \{a + b \mid a \in X, b \in Y\}$ contains $n^2$ distinct integers or whether there are duplicates.

Show that if the $X + Y$ problem can be solved in time $O(n^{2-\epsilon})$ for some $\epsilon > 0$, then 3SUM can be solved in time $O(n^{2-\delta})$ for some $\delta > 0$.

Exercise 3 Hitting Set Problem: Given sets $S_1, \ldots, S_n, T_1, \ldots, T_n \subseteq \{1, \ldots, d\}$, determine whether there is a set $S_i$ that intersects every set $T_j$ (in this case $S_i$ is called a “hitting set”).

Clearly this problem can be solved in time $O(n^2d)$. The Hitting set Hypothesis (HSH) states that this problem cannot be solved in time $O(n^{2-\epsilon} \cdot \text{poly}(d))$.

Prove that HSH implies OVH.

Exercise 4 ZeroTriangle: Given a weighted directed graph $G = (V, E, w)$ with edge weights $w : E \to \{-n^c, \ldots, n^c\}$ (for some large constant $c > 0$), determine whether there are three vertices $i, j, k$ such that $w(i,j) + w(j,k) + w(k,i) = 0$ holds.

Clearly this problem can be solved in time $O(n^3)$. Prove that if ZeroTriangle can be solved in time $O(n^{3-\epsilon})$ (for some $\epsilon > 0$) then:

a) APSP can be solved in time $O(n^{3-\delta})$ (for some $\delta > 0$), and

b) 3SUM can be solved in time $O(n^{2-\delta})$ (for some $\delta > 0$).

Completion of Lecture:

Exercise 5 Prove that MaxSubmatrix is equivalent to APSP under subcubic reductions, i.e., complete the partial proof from the lecture.

Exercise 6 Construct a $k$-sum-free set $S \subseteq \{1, \ldots, U\}$ of size $n$ over universe $U = n^{1+\epsilon} k^{O(1/\epsilon)}$, i.e., work out the details of the construction sketched in the lecture.