Learning and Games, day 4
Price of Anarchy and Game Dynamics

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Learning and Games

Price of Anarchy and Game Dynamics

Day 4: Can learning do better than worst Nash?
Main question:
Quality of Selfish outcome

Selfish outcome = result of Learning behavior

Our Question: quality of learning outcomes?
which correlated equilibrium do users coordinate on?

Answer: depends on which learning...

Theorem: $\forall$ correlated equilibrium is the limit point of no-regret play
Correlated eq. = learning outcome?

**Proof:** Intelligent designer algorithm

Take a coarse correlated equilibrium

assume probabilities $p$ rational

Design a sequence of moves that has desired distribution $(\frac{1}{2};\frac{1}{4},\frac{1}{4},0)$

Sequence

$(1,1), (2,1), (1,1), (1,2)$

Repeat!
Correlated eq. = learning outcome?

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Sequence
(1,1), (2,1), (1,1), (1,2)
Repeat!

Intelligent designer algorithm

- Follow the designed sequence as long as all other players do.
- If anyone deviates: switch to smoothed fictitious play

This is no regret!
Quality of Learning Outcome

Price of Anarchy [Koutsoupias-Papadimitriou’99]

\[ PoA = \max_{a \text{ Nash}} \frac{\text{cost}(a)}{\text{Opt}} \]

Assuming no-regret learners in fixed game: [Blum, Hajiaghayi, Ligett, Roth’08, Roughgarden’09]

\[ PoA = \lim_{T \to \infty} \frac{\sum_{t=1}^{T} \text{cost}(a^t)}{T \text{Opt}} \]

[Lykouris, Syrgkanis, T. 2016] dynamic population

\[ PoA = \lim_{T \to \infty} \frac{\sum_{t=1}^{T} \text{cost}(a^t, v^t)}{\sum_{t=1}^{T} \text{Opt}(v^t)} \]

where \( v^t \) is the vector of player types at time \( t \)
A Game with Bad PoA

Personal objective: minimize

\[ c_p(f) = \text{sum of costs of edges along } P \text{ (wrt. flow } f) \]

Overall objective:

\[ C(f) = \text{total cost of a flow } f: = \sum_e f_e \cdot c_p(f_e) \]

= - social welfare
or total/average cost

Economy of scale
Cost-sharing: a bad example: $c_e(x) = c_e/x$

Claim: this is the worst case
Maybe Best Nash is good?

We know price of anarchy is bad, but *Price of Stability* is better.

**Price of Stability** = \[
\frac{\text{cost of best selfish outcome}}{\text{“socially optimum” cost}}
\]

**Theorem** [Anshelevich, Dasgupta, Kleinberg, Tardos, Wexler, Roughgarden FOCS’04]
Price of Stability is at most \( H_k = O(\log k) \) for \( k \) players, while price of anarchy is at most \( k \)
Selfish Outcome= non-cooperative?

Nash equilibrium: non-cooperative outcome
  • Current strategy “best response” for all players
  • no single user has incentive to deviate

How about groups of players?

Strong Nash equilibrium: no group of players has incentive to deviate [Aumann’59]
Cooperative game?

We can use: **Strong Nash equilibrium**

• No subset players can coordinate a deviation and improve for every player in the set

[Epstein, Feldman, Mansour EC’07]

the strong price of anarchy is $H_k = O(\log k)$

(but strong Nash may not exists...)
Open problems

Is there a simple dynamic that leads to such better outcomes?

• Learning or best response (random best response?) from a random start? Or from users arriving one-by-one?

• What is a cooperative dynamic?
Illustrative Example:

All noes 1... k want to connect to terminal t
Example:

All nodes 1...k want to connect to terminal t

\[ \text{cost(OPT)} = 1 + \varepsilon \]
Example:

All nodes 1... k want to connect to terminal t

cost(OPT) = 1+ε

...but not a NE:
player k pays \((1+\varepsilon)/k\), could pay \(1/k\)
Example:

All nodes 1... k want to connect to terminal t

so player k would deviate
Example:

All nodes 1... k want to connect to terminal t

now player k-1 pays \((1+\varepsilon)/(k-1)\), could pay \(1/(k-1)\)
Example:

All noes $1 \ldots k$ want to connect to terminal $t$

so player $k-1$ deviates too
Example:

All nodes 1... k want to connect to terminal t

Continuing this process, all players defect.

This is a NE!
(the only Nash)

cost = 1 + \frac{1}{2} + \ldots + \frac{1}{k}

In fact, a strong Nash

Price of Stability is $H_k = \Theta(\log k)$!
But: strong Nash $\exists$?

[Epstein, Feldman, Mansour EC’07]  
the strong price of anarchy is $O(\log k)$

But $\exists$?: No!!

Nash unique: cost of 5 each

It is not strong! As there is a solution better for both  

cost of 4 each  
It’s a “prisoner dilemma”

$\Rightarrow$ no strong Nash exists $\exists$
Proof idea: congestion games have potentials

\[ \Phi(f) = \sum_e (c_e(1)+\ldots+c_e(f_e)) = \sum_e \Phi_e \]

[in non-atomic game \( \Phi=\sum_e \int_0^{f_e} c_e(\xi) d\xi \)]

Theorem (Rosenthal) if player \( i \) moves from path \( P \) to a new path \( Q \)
Improving her cost by \( \Delta \), then potential decreases by \( \Delta \)

Proof: if player \( i \), was using path \( P \) and now leaves the game, \( \Phi(f) \)
decreases by \( \sum_{e \in P} c_e(f_e) \), which is player \( i \)'s cost.
Now she re-enters on path \( Q \). Use the same argument.
Congestion games are potential games

This implies a few useful things

• Nash = local minima of potential $\Phi$

• Repeated best response leads to a Nash equilibrium: decreases potential $\Phi$

• Learning also leads to Nash equilibria (not to correlated equilibria!)
Nash for 2-ball & 2-bin

A correlated eq. $1/3$–$1/3$–$1/3$

The set of correlated equilibria

Mixed Nash

Pure Nash

$\mathbf{c}_e(x) = x$

$\begin{array}{c|c}
2 & 1 \\
1 & 2 \\
\end{array}$
Using potential $\Phi$...

• Consider the Nash with minimum value of $\Phi$
• This Nash has,
  $$\Phi(\text{Nash}) < \Phi(\text{OPT}).$$

Suppose that we also know for any solution
  $$\Phi \leq \text{cost} \leq A \Phi$$

$$\rightarrow \text{cost(\text{Nash})} \leq A \Phi(\text{Nash}) \leq A \Phi(\text{OPT}) \leq A \text{cost(\text{OPT})}.$$  
$$\rightarrow$$ There is a good Nash!
Results for Cost sharing

proof:
Recall: $\Phi(f) = \sum_e (c_e(1)+...+c_e(f_e)) = \sum_e \Phi_e$

$f_e \leq k$ users on edge $e$ then

- true cost is $c_e$ with any $>0$ users
- Potential is $\Phi_e = c_e + c_e/2 + c_e/3 + ... + c_e/f_e$
  \[ \leq c_e \cdot (1 + 1/2 + 1/3 + ... + 1/k) = c_e H_k \]

- cost $\leq \Phi \leq \text{cost} \cdot H_k$

\[ \rightarrow \text{Nash optimizing } \Phi \text{ cost at most } H_k \text{ above the optimum} \]
Example

All nodes 1... k want to connect to terminal t

Continuing this process, all players defect.

This is a NE! (the only Nash)

cost = 1 + \frac{1}{2} + \cdots + \frac{1}{k}

In fact, a strong Nash

Price of Stability is $H_k = \Theta(\log k)!$
**Strong Price of Anarchy?**

\( SE = \) strong Nash, \( Opt \)

As a group not all players want to move to \( Opt \):

\( \Rightarrow \) There exists player, say last player \( k \), that is better off in current solution

\( \Rightarrow \) \( \text{cost}_k(SE) \leq \text{cost}_k(Opt) \)

Consider remaining \( k-1 \) players.

\( \text{Opt}_{k-1} = \) Opt restricted to remaining \( k-1 \) players

As a group the remaining \( k-1 \) players also don't want to move to \( \text{Opt}_{k-1} \) \( \Rightarrow \) there is a player, say \( k-1 \)

\( \text{Cost}_{k-1}(SE) \leq \text{cost}_{k-1}(\text{Opt}_{k-1}) \)
Strong Price of Anarchy

**SE** = strong Nash, **Opt**,

Continue...

**Opt** \(i\) = Opt restricted to remaining \(i\) players.

We get: \(\text{cost}_i(\text{SE}) \leq \text{cost}_i(\text{Opt}_i)\)

**Lemma:** In potential games: \(\text{cost}_i(\text{Opt}_i) = \Phi(\text{Opt}_i) - \Phi(\text{Opt}_{i-1})\)

**Proof:** consider first \(i\) players only, and selfish move of player \(i\) of “not playing”:

- **Cost** to player \(i\): \(\text{cost}_i(\text{Opt}_i)\)
- **potential change** \(\Phi(\text{Opt}_i) - \Phi(\text{Opt}_{i-1})\)
Strong Price of Anarchy

\( SE = \text{strong Nash, Opt,} \)

\( \text{Opt}_i = \text{Opt restricted to first i} \)

set 1...i doesn’t want to move

\( \text{cost}_i(SE) \leq \text{cost}_i(\text{Opt}_i) \)

Potential game: \( \text{cost}_i(\text{Opt}_i) = \Phi(\text{Opt}_i) - \Phi(\text{Opt}_{i-1}) \)

We get: \( \text{cost}_i(SE) \leq \text{cost}_i(\text{Opt}_i) = \Phi(\text{Opt}_i) - \Phi(\text{Opt}_{i-1}) \)

\( \sum_i \text{cost}_i(SE) \leq \sum_i \Phi(\text{Opt}_i) - \Phi(\text{Opt}_{i-1}) = \Phi(\text{Opt}) \)

In cost-sharing game \( \Phi(\text{Opt}) \leq H_k \text{cost}(\text{Opt}) \)
Dynamic with cooperation?

**Cooperation:** group of users deviate together to improve their welfare

Cooperative game theory...

- No great model for outcome for most games
- Strong Nash: outcome when collusion is not useful.
- But what happens when no such outcome exists: collusion is useful?
- **Bargaining:** agreement when everyone colludes
  - different bargaining “games” characterized by axioms
Does learning lead to better Nash?

- Idea 1: with a uniformly random start?

- Idea 2: with each player arriving one-by-one, while others are repeatedly best responding
  - Charikar, Karloff, C. Mathieu, J. Naor, Saks, SPAA’08: $O(\log^3 n)$ PoA if
    - single source
    - all players arrive before any best response
  - idea: arrival phase is $\approx$ online Steiner tree, then use potential function
Outcome of Multiplicative Weights

Theorem: R. Kleinberg-Piliouras-Tardos multiplicative weight like processes with small $\epsilon$ converge to pure Nash in almost all congestion games.

Recall
In congestion games learning converges a Nash (decreases potential).

Which one? Uniform random has cost $(1 - \frac{1}{e}) nc \approx 0.63 nc$, while $\text{Opt} = c$.
Continuous limit of multiplicative weights

Multiplicative weight with $\varepsilon \sim 0$:
- probability of playing action $x$ is $p_x^t \leftarrow w_x^t / \sum_i w_i^t$
- Update $w_x^{t+1} \leftarrow w_x^t \alpha c_i(x,s_i^t)$

Limit as update gets smaller.

Limit as update get slower $\alpha = 1 - \varepsilon$
- $\lim_{\varepsilon \to 0} \frac{p_x^{t+1} - p_x^t}{\varepsilon} = p_x^t (\sum_y p_y^t c_i(y,s_i^t) - c_i(x,s_i^t))$

Limit $\Rightarrow$ replicator dynamic: dynamical system

$$p_x = p_x (\sum_y p_y c_i(y,s_{-i}) - c_i(x,s_{-i}))$$
What are weakly stable points

Weakly stable for ODE= neighborhood no direction has pull away from fixed point.
Stable points in 2-ball 2-bin

Mixed Nash

Pure Nash

Non-Nash

$e_e(x) = x$
What are weakly stable points

Weakly stable for ODE = neighborhood no direction has pull away from fixed point.

Lemma Weakly stable fixed points are Nash equilibria

Why?
fixed point: $p_e > 0 \Rightarrow \exp(\text{cost}) = \text{cost}(e)$
If $\exp(\text{cost}) > \text{cost}(e)$ (for some $p_e = 0$)
$\Rightarrow$ not stable
Example: 2-balls 2-bins

Weakly stable for ODE= neighborhood no direction has pull away from fixed point.

Fact: Mixed Nash in 2-balls 2-bins game not stable
Learning as a symmetry breaking

Simple case
2 balls & 2 bins

Players choose different bins ⇒

• they “learn” that chosen bin better
  other bin would have bigger congestion

• reinforcing the decision
Weakly Stable Nash?

Weakly stable in games: each player remains indifferent between the strategies when one other player chooses a fixed strategy.

Example: balls & bins
Weakly stable $\Rightarrow$ at most one random player in each bin.

Random Nash stable:
1 ball and 2 bins.

$\mathbf{c}_e(x) = x$
Summary from this week

simple games and variants:
• matching pennies,
• coordination,
• prisoner’s dilemma,
• Rock-paper-scissor

Learning algorithms
• Fictitious play, and smoothed versions

No-regret as outcome of learning or as a behavioral model

Price of Anarchy and learning outcomes (including changing environments) in
• Congestion games, such as traffic routing
• Auction games

Learning in multi-item auctions is hard,
Alternate learning we can do instead
Best Nash in congestion games, and what learning does in such games