ADFOCS 2017

multilinear world

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introduction
multilinear polynomials

determinant

\[ det_n(X) = \sum_{\pi \in S_n} \text{sign}(\pi) \prod_{i \in [n]} X_{i,\pi(i)} \]

permanent

\[ perm_n(X) = \sum_{\pi \in S_n} \prod_{i \in [n]} X_{i,\pi(i)} \]

symmetric polynomials

\[ S_{n,d}(X) = \sum_{T \subseteq [n]: |T| = d} \prod_{i \in T} x_i \]
multilinear complexity [Nisan-Wigderson]

a polynomial is multilinear if individual degrees are at most 1

multilinear circuit\(^1\)

\[ v = v_1 \times v_2 \Rightarrow \text{var}(v_1) \cap \text{var}(v_2) = \emptyset \]

multilinear ABP: no variable appears twice on \( a \rightarrow b \) paths

a monotone device for multilinear polynomial is multilinear

what are multilinear complexities of multilinear polynomials?

\(^1\)syntactic
multilinear world

all simulations preserve multilinearity, except depth 3
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[Raz, Raz-Y]

circuits are super-poly stronger than formulas
multilinear world

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[Dvir-Malod-Perifel-Y]

ABPs are super-poly stronger than formulas
multilinear world

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ABPs are super-poly stronger than formulas

[Raz-Y]
circuits of depth* $d + 1$ are super-poly stronger than depth $d$
lower bounds

[Raz]

multilinear formulas for $\text{det}_n$ or $\text{perm}_n$ are of size $n^{\Omega(\log n)}$

[Shpilka-Raz-Y]

$\tilde{\Omega}(n^{4/3})$ multilinear circuit-size lower bound

[Raz-Y]

depth $d$ multilinear circuits for $\text{det}_n$ or $\text{perm}_n$ are of size $2^n^{\Omega(1/d)}$
lower bounds
I. identify a weakness of multilinear formulas

II. exploit it, preferably combinatorially

wish: avoid algebra and argue combinatorially
Lemma

If $f$ is $n$-variate multilinear formula-size $s$ then

$$f = \sum_{i=1}^{s} g_i$$

where each $g_i$ is log-product:

$$g_i = g_{i,1} g_{i,2} \cdots g_{i,t}$$

with $t = \Omega(\log n)$ and there is a partition of $X$ to $X_{i,1}, \ldots, X_{i,t}$ so that

$$|X_{i,j}| \geq n^{1/2}$$

and

$$\text{var}(g_{i,j}) \subseteq X_{i,j}$$
to exploit weakness find a “measure” that is

- small on log-product
- sub-additive
- large for some polynomial of interest
exploiting weakness

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[Nisan] the partial derivative matrix

[Raz] random partitions
given $f \in \mathbb{F}[Y, Z]$ define a matrix $M = M_f$ by

$$M_{p,q} = \text{coefficient of } pq \text{ in } f$$

where $p, q$ are monomials in $Y, Z$
partitions

A polynomial \( f \in \mathbb{F}[X] \) is a vector not a matrix.

Given \( \pi : X \rightarrow Y \cup Z \), the polynomial \( f_{\pi}(Y, Z) = f(\pi(X)) \) comes with the matrix \( M_{\pi} = M_{f_{\pi}} \).

There are many such matrices for \( f \); you can choose one after seeing the alleged formula.
partitions

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given $\pi : X \rightarrow Y \cup Z$ the polynomial

$$f_\pi(Y, Z) = f(\pi(X))$$

comes with the matrix $M_\pi = M_{f_\pi}$
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A polynomial \( f \in \mathbb{F}[X] \) is a vector not a matrix.

given \( \pi : X \to Y \cup Z \) the polynomial

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f_{\pi}(Y, Z) = f(\pi(X))
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comes with the matrix \( M_{\pi} = M_{f_{\pi}} \)

There are many such matrices for \( f \)

can choose one **after** seeing the alleged formula.
f has **full-rank** if for every partition π of X to two parts of equal size the partial derivative matrix $M_\pi$ has full rank
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**Theorem [Raz]**

if \( f \) has full-rank then every multilinear formula for \( f \) has size at least \( n^{\Omega(\log n)} \)
properties of rank

let \( f \in \mathbb{F}[Y, Z] \) be multilinear with \(|Y| + |Z| = n\)

1. if \( f = gh \) then \( M_f = M_g \otimes M_h \) and
   \[
   \text{rank}(M_f) = \text{rank}(M_g) \cdot \text{rank}(M_h)
   \]

2. if \( f = g + h \) then \( M_f = M_g + M_h \) and
   \[
   \text{rank}(M_f) \leq \text{rank}(M_g) + \text{rank}(M_h)
   \]

3. \[
   \text{rank}(M_f) \leq 2^{\min\{|Y|, |Z|\}} \leq 2^{(n-\Delta)/2}
   \]

where \( \Delta = ||Y| - |Z|| \)
lemma (random partitions)

$X$ is partitioned to $X_1, \ldots, X_t$ each of size $n_j \geq n^{1/2}$

choose uniformly at random a bijection

$$\pi : X \rightarrow Y \cup Z$$

where $|X| = n$ and $|Y| = |Z| = n/2$

then

$$\Pr \left[ \Delta_j < n^{1/100} \text{ for all } j \right] < n^{-t/1000}$$

where

$$\Delta_j = \left| |Y_j| - |Z_j| \right|$$

and $Y_j, Z_j$ come from $\pi(X_j)$
intuition

$X$ is partitioned to $X_1, \ldots, X_t$ each of size $n_j \geq n^{1/2}$

$\pi : X \to Y \cup Z$ is random

$\Delta_j = ||Y_j| - |Z_j||$

idea

1. “independence”

$$\Pr \left[ \Delta_j < n^{1/100} \text{ for all } j \right] \approx \prod_j \Pr \left[ \Delta_j < n^{1/100} \right]$$

2. “anti-concentration”

$$\Pr \left[ \Delta_j < n^{1/100} \right] \lesssim \frac{2n^{1/100}}{n_j^{1/2}} \leq n^{-1/1000}$$
LB: the calculation

write \( f = \sum_{i=1}^{s} g_i \) where \( g_i \) is log-product with \( s < n^{\log(n)/1000} \)

choose a partition \( \pi \) at random and set \( M = M_\pi \)
LB: the calculation

write \( f = \sum_{i=1}^{s} g_i \) where \( g_i \) is log-product with \( s < n^{\log(n)/1000} \)

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\[
1 = \Pr \left[ \text{rank}(M) = 2^{n/2} \right] = \Pr \left[ \text{rank} \left( \sum_i M_i \right) = 2^{n/2} \right]
\]
\[
\leq \Pr \left[ \sum_i \text{rank}(M_i) \geq 2^{n/2} \right] \leq \Pr \left[ \exists i \text{ rank}(M_i) \geq 2^{n/2 - \log s} \right]
\]
\[
\leq \sum_i \Pr \left[ \text{rank}(M_i) \geq 2^{n/2 - \log s} \right]
\]
\[
= \sum_i \Pr \left[ \prod_j \text{rank}(M_{i,j}) \geq 2^{n/2 - \log s} \right]
\]
\[
\leq \sum_i \Pr \left[ |\Delta_{i,j}| < n^{1/100} \text{ for all } j \right] \leq s \cdot n^{-\Omega(\log n)}
\]
if $f$ has multilinear formula of size $s$

**weakness:** write $f$ as a sum of $s$ log-product polynomials

**randomness:** if $s$ is small then there is a partition that makes all log-products of “low rank”

**full rank:** $f$ has full-rank so $s$ is large
full-rank polynomials
both $det_n$ and $perm_n$ are full-rank with respect to some “rich enough” family of partitions
separating circuits and formulas [Raz-Y]

let $X = \{x_1, \ldots, x_n\}$ and $Y$ be extra variables

for $b - a = 1$ define

$$p_{a,b} = x_a + x_b$$

and for $b - a$ odd inductively define

$$p_{a,b} = y_{a,b}(x_a + x_b)p_{a+1,b-1} + \sum_{k:k-a \text{ odd}} y_{a,b,k}p_{a,k}p_{k+1,b}$$
separating circuits and formulas [Raz-Y]

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properties

1. $p_{a,b}$ has a multilinear circuit of size $\text{poly}(n)$
2. $p_{a,b}$ is full-rank with respect to $X_{a,b}$ over $\mathbb{F}(Y)$
separating ABPs and formulas [Dvir-Malod-Perifel-Y]

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$$p_{a,b} = y_{1,a,b}p_{a+1,b-1}(x_a + x_b) + y_{2,a,b}p_{a+2,b}(x_a + x_{a+1}) + y_{3,a,b}p_{a,b-2}(x_{b-1} + x_b)$$

where addition is modulo $n$
separating ABPs and formulas [Dvir-Malod-Perifel-Y]

for $b - a = 1$ define

$$p_{a,b} = x_a + x_b$$

and for $b - a$ odd inductively define

$$p_{a,b} = y_{1,a,b} p_{a+1,b-1} (x_a + x_b) + y_{2,a,b} p_{a+2,b} (x_a + x_{a+1}) + y_{3,a,b} p_{a,b-2} (x_{b-1} + x_b)$$

where addition is modulo $n$

**properties**

1. $p_{a,b}$ has a multilinear ABP of size $poly(n)$
2. $p_{a,b}$ is **not** full-rank with respect to $X_{a,b}$ over $\mathbb{F}(Y)$
3. formula lower bound can still be proved
lower bounds for ABPs?

no strong lower bound for multilinear ABPs

conjecture

if $f$ has full-rank then any multilinear ABP for $f$ has super-poly size
summary
many natural multilinear polynomials

multilinear devices are a natural way to compute them

know how to prove strong lower bounds for multilinear formulas

grading multilinear polynomials by number of variables

what about ABPs or circuits?