ADFOCS Lectures

- Asynchronous Crash-Prone Distributed Computing
- \checkmark\ Locality in Distributed Network Computing
- Congestion-Prone Distributed Network Computing
- Other Aspects of Distributed Computing
Various Models

Shared Memory  Message Passing  Synchronous  Asynchronous

Failures: crash, transient, Byzantine, etc.
Networks

Two major technological constraints:

- Latency / Locality
- Bandwidth / Information
LOCAL Model

- Each process is located at a node of a network modeled as an $n$-node graph ($n = \#\text{processes}$)
- Each process has a unique ID in $\{1,\ldots,n\}$
- Computation proceeds in synchronous rounds during which every process:
  1. Sends a message to each neighbor
  2. Receives a message from each neighbor
  3. Performs individual computation (same algorithm for all nodes)
Complexity = #rounds

Lemma If a problem $P$ can be solved in $t$ rounds in the LOCAL model by an algorithm $A$, then there is a $t$-round algorithm $B$ solving $P$ in which every node proceeds in two phases:

Phase 1. Gather all data in the $t$-ball around it

Phase 2. Compute the solution
Graph problems

Vertex coloring

Independent set
(Δ+1)-coloring

Δ = maximum node degree of the graph

(Δ+1)-coloring = assign colors to nodes such that every pair of adjacent nodes are assigned different colors.

Lemma Every graph is (Δ+1)-colorable

Theorem (Brooks, 1941)
Every graph G is Δ-colorable, unless G is a complete graph, or an odd cycle.
Maximal Independent Set (MIS)

- Maximal, not maximum!
Roadmap

1. Deterministic algorithms

2. Randomized algorithms

3. Strong links between deterministic and randomized algorithms
Deterministic Algorithms
3-coloring the $n$-node cycle $C_n$

How many rounds for 3-coloring the $n$-node cycle?
Round complexity of 3-coloring $C_n$

**Theorem** (Cole and Vishkin, 1986) There exists an algorithm for 3-coloring $C_n$ performing in $O(\log^*n)$ rounds.

Iterated logarithms:

- $\log^{(0)} x = \log x \quad \log^{(k+1)} x = \log \log^{(k)} x$
- $\log^* x = \text{smallest } k \text{ such that } \log^{(k)} x < 1$
- $\log^* 10^{100} = 5$

**Theorem** (Linial, 1992) Any 3-coloring algorithm for $C_n$ performs in $\Omega(\log^*n)$ rounds.  

Dijkstra Prize 2013
Cole-Vishkin Algorithm

Initial color = ID
Express colors in binary

Assume: n is known, and consistent sens of direction

new = (position, bit) = (5, 1) = 1011

(p', b')

(p, b)

p ≠ p' ⇒ proper coloring
p = p' ⇒ b ≠ b' ⇒ proper coloring
Number of iterations

• $k$-bit colors $\rightarrow$ new colors on $\lceil \log_2 k \rceil + 1$ bits

• $\log^* n + O(1)$ rounds to reach colors on 3 bits

• 8 colors down to 3 colors in 5 rounds

• Total number of rounds = $\log^* n + O(1)$
Speeding up the Algorithm

- Every node can simulate 2 rounds in just 1 round
- left round + right round $\implies$ implemented in 1 round
- Total number of rounds $= \frac{1}{2} \log^* n + O(1)$
Linial’s Lower Bound

t-round algorithm

every node $x$ decides as a function $A$ applied to $B_t(x)$ where $B_t(x) = (g_t, g_{t-1}, \ldots, g_1, x, d_1, \ldots, d_{t-1}, d_t)$
Configuration Graph $G_{t,n}$

vertices $= \{(g_t, \ldots, g_1, x, d_1, \ldots, d_t) \in \{1, \ldots, n\}^{2t+1}\}$

edges $= \{(g_t, \ldots, g_1, x, d_1, \ldots, d_t) \rightarrow (g_{t-1}, \ldots, g_1, x, d_1, \ldots, d_t, d_{t+1})\}$

1. $t$-round 3-coloring algorithm for $C_n \Rightarrow \chi(G_{t,n}) \leq 3$

2. $t < \frac{1}{2} \log^* n - O(1) \Rightarrow \chi(G_{t,n}) > 3$
Step 1

Lemma  t-round c-coloring algo for $C_n \Rightarrow \chi(G_t, n) \leq c$

Proof  Algo $\mathcal{A} \Rightarrow$ vertex $(g_t, \ldots, g_1, x, d_1, \ldots, d_t)$ colored

$\mathcal{A} (g_t, \ldots, g_1, x, d_1, \ldots, d_t)$

Coloring is proper as

$(g_t, \ldots, g_1, x, d_1, \ldots, d_t)$ and $(g_{t-1}, \ldots, g_1, x, d_1, \ldots, d_{t+1})$

can appear as view of $x$ and $d_1$ in some instances of ID assignment to the nodes of the ring.
Corollary (Linial, 1992) For $n$ even, 2-coloring $C_n$ requires $\Omega(n)$ rounds.

Proof Assume $t$ rounds, with $t \leq n/2 - 2 \Rightarrow 2t+1 \leq n-3$.

1. $(x_1, x_2, \ldots, x_{2t+1})$
2. $(x_2, \ldots, x_{2t+1}, y)$
3. $(x_3, \ldots, x_{2t+1}, y, z)$
4. $(x_4, \ldots, x_{2t+1}, y, z, x_1)$
5. $(x_5, \ldots, x_{2t+1}, y, z, x_1, x_2)$
   
   $\vdots$

$2t+1. \ (x_{2t+1}, y, z, x_1, \ldots, x_{2t-2})$
$2t+2. \ (y, z, x_1, \ldots, x_{2t-2}, x_{2t-1})$
$2t+3. \ (z, x_1, \ldots, x_{2t-1}, x_{2t})$

odd cycle

$\chi(G_{t,n}) > 2$
Step 2

**Lemma** \( t < \frac{1}{2} \log^* n - O(1) \Rightarrow \chi(G_{t,n}) > 3 \)

Proof is technical (uses line graphs)\(^1\)

But worth reading!

\(^1\)Other proofs use Ramsey theory.
A simpler proof of Linial’s lower bound

Proof (Laurinharju & Suomela, 2014)

\( \mathcal{A} \) is a \( k \)-ary \( c \)-coloring function if

1. \( \mathcal{A}(x_1, x_2, \ldots, x_k) \in \{1, 2, \ldots, c\} \) for all \( 1 \leq x_1 < x_2 < \ldots < x_k \leq n \)
2. \( \mathcal{A}(x_1, x_2, \ldots, x_k) \neq \mathcal{A}(x_2, x_3, \ldots, x_{k+1}) \) for all \( x_k < x_{k+1} \leq n \)

Claim 0. \( t \)-tound algorithm \( \mathcal{A} \) for 3-coloring \( C_n \)

\( \Rightarrow \mathcal{A} \) is \( (2t+1) \)-ary 3-coloring function

Claim 1. If \( \mathcal{A} \) is a 1-ary \( c \)-coloring function then \( c \geq n \).
Claim 2. If $\mathcal{A}$ is a k-ary c-coloring function, then there is a $(k-1)$-ary $2^c$-colouring function $\mathcal{B}$.

\[
\mathcal{B}(x_1, x_2, \ldots, x_{k-1}) = \{ \mathcal{A}(x_1, x_2, \ldots, x_{k-1}, x_k) : x_k > x_{k-1} \}
\]

For contradiction, let $1 \leq x_1 < x_2 < \ldots < x_k \leq n$ with
\[
\mathcal{B}(x_1, x_2, \ldots, x_{k-1}) = \mathcal{B}(x_2, \ldots, x_{k-1}, x_k)
\]

Let $c = \mathcal{A}(x_1, x_2, \ldots, x_{k-1}, x_k)$.

$\Rightarrow c \in \mathcal{B}(x_1, x_2, \ldots, x_{k-1}) \Rightarrow c \in \mathcal{B}(x_2, \ldots, x_{k-1}, x_k)$

$\Rightarrow \exists x_{k+1} > x_k : c = \mathcal{A}(x_2, \ldots, x_k, x_{k+1}) \Rightarrow \mathcal{A}$ is not k-ary c-coloring function.
**Theorem** Any $3$-coloring algorithm for $C_n$ performs in $\Omega(\log^* n)$ rounds.

**Proof** Let $\mathcal{A}$ be a $t$-tound algorithm for $3$-coloring $C_n$

$\Rightarrow \mathcal{A}$ is $(2t+1)$-ary $3$-coloring function (by Claim 0)

$\Rightarrow \mathcal{A}$ is $(2t)$-ary $2^3$ -coloring function (by Claim 2)

$\Rightarrow \mathcal{A}$ is $(2t-1)$-ary $2^{(2)\cdot 3}$-coloring function

$\Rightarrow \mathcal{A}$ is $(2t-2)$-ary $2^{(3)\cdot 3}$-coloring function

$\vdots$

$\Rightarrow \mathcal{A}$ is $(1)$-ary $2^{(2t)\cdot 3}$-coloring function

$\Rightarrow 2^{(2t)\cdot 3} \geq n$ (by Claim 1)

$\Rightarrow t \geq \frac{1}{2} \log^* n - 1.$

$\square$
(Δ+1)-coloring arbitrary graphs

- Best lower bound (Linial, 1992)
  \[ \Omega(\log^* n) \] rounds
- Best upper bound (Panconesi & Srinivasan, 1992)
  \[ 2^{O(\sqrt{\log n})} \] rounds

Gap open for a quarter of a century!
(Δ+1)-coloring arbitrary graphs

Best lower bound (Linial, 1992): $\Omega(\log^* n)$ rounds

Best upper bound (Panconesi & Srinivasan, 1992): $2^{O(\sqrt{\log n})}$ rounds

Gap open for a quarter of a century!

(Δ+1)-coloring in $\log^{O(1)} n$ rounds!

V. Rozhon and M. Ghaffari (2019)
Complexity as $f(n) + g(\Delta)$

**Theorem** (Linial, 1992)
There is a $(\Delta + 1)$-coloring algorithm performing in $O(\log^* n) + \tilde{O}(\Delta^2)$ rounds.

**Theorem** (F., Heinrich, Kosowski, 2016)
There is a $(\Delta + 1)$-coloring algorithm performing in $O(\log^* n) + \tilde{O}(\sqrt{\Delta})$ rounds.
**O(Δ²)-coloring**

**Theorem** (Linial, 1992) \( O(Δ²) \)-coloring in \( \log^* n + O(1) \) rounds

**Lemma** For all \( k > Δ ≥ 2 \), there exists \( J = \{S_1, \ldots, S_k\} \) where

\[
S_i \subseteq \{1, \ldots, 5 \lceil Δ^2 \log k \rceil\} \quad \text{for} \; i=1, \ldots, k
\]

such that, for every \( Δ+1 \) sets \( S_{i0}, S_{i1}, \ldots, S_{iΔ} \) in \( J \), we have

\[
S_{i0} \not\subseteq U_{j=1, \ldots, Δ} S_{ij}.
\]

**Algorithm:** Init: \( k = n \) and color(u) = ID(u)

Each round: color range \([1, k]\) reduced to \([1, 5 \lceil Δ^2 \log k \rceil]\)

\[
\text{color}(u) = c \quad \Rightarrow \quad u \; \text{has set} \; S_c
\]

New color: smallest \( x \in S_c \setminus U_{i=1, \ldots, Δ} S_{\text{color}(v_i)} \).
Locally Iterative Algorithm

**Theorem** [L. Barenboim, M. Elkin, U. Goldenberg (2017)]
There exists a locally iterative algorithm for \((\Delta + 1)\)-coloring, performing in \(O(\log^* n + \Delta)\) rounds.

**Proof.** Compute \(O(\Delta^2)\)-coloring in \(\log^* n + O(1)\) rounds.
Assume for simplicity a \((\Delta + 1)^2\)-coloring with \(\Delta + 1 = p\) prime.
Represent color \(c_0(v) = (a_v, b_v)\) where \(a_v, b_v \in \text{GF}(p)\).

- if \(\nexists u \in N(v), \text{ with } b_u = b_v\) then \(v\) adopts \((0, b_v)\) as final color;
- otherwise, \(v\) recolors itself as \((a_v, b_v + a_v)\).

The following two properties hold:
- Recoloring preserves proper coloring
- After \(2p + 1 = 2(\Delta + 1) + 1\) rounds, all nodes have finalized their color.
Locally Checkable Labeling

Let \( \mathcal{F}_\Delta \) be the set of all (connected) graphs with maximum degree \( \Delta \).

**Definition** (Naor and Stockmeyer, 1995) An LCL in \( \mathcal{F}_\Delta \) is specified by a finite set of labels, and a finite set of labeled balls with maximum degree \( \Delta \), called **good balls**.

**Examples:**
- \( k \)-coloring, \( k \)-edge-coloring
- maximal independent set (MIS)
- maximal matching
- Etc.

Focus is on LCL tasks solvable sequentially by a greedy algorithm selecting nodes in arbitrary order, like, e.g., \( k \)-coloring for \( k \geq \Delta + 1 \).
Maximal Independent Set

- $(\Delta + 1)$-coloring $\Rightarrow$ MIS in $\Delta$ rounds by maximizing $\{1\}$

- MIS $\Rightarrow$ $(\Delta + 1)$-coloring by simulation
Claim 1. At most one node of each clique in the MIS

Claim 2. At least one node of each clique in the MIS

Color = index of node in the MIS
Line Graphs

**Definition** The line graph of a graph $G$ is the graph $L(G)$ such that

- $V(L(G)) = E(G)$
- $\{e,e'\} \in E(L(G)) \iff e$ and $e'$ are incident in $G$
Four classical problems

- Vertex Coloring
- Edge Coloring

MIS

MIS on line graph

Maximal Matching

- coloring on line graph

- Edge Coloring
# Round Complexity

<table>
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<tr>
<th></th>
<th>MIS</th>
<th>(Δ+1)-coloring</th>
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<tbody>
<tr>
<td><strong>Deterministic</strong></td>
<td>$2^{\sqrt{\log(n)}}$</td>
<td>$2^{\sqrt{\log(n)}}$</td>
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<tr>
<td><strong>Randomized</strong></td>
<td>$2^{\sqrt{\log\log(n)}} + O(\log \Delta)$</td>
<td>$2^{\sqrt{\log\log(n)}}$</td>
</tr>
<tr>
<td><strong>Maximal Matching</strong></td>
<td><strong>(2Δ-1)-edge-coloring</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Deterministic</strong></td>
<td>$O(\log^3 n)$</td>
<td>$O(\log^6 n)$</td>
</tr>
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<td></td>
<td></td>
<td>Ghaffari, Harris, Kuhn (2018)</td>
</tr>
<tr>
<td><strong>Randomized</strong></td>
<td>$O(\log^3 \log n) + O(\log \Delta)$</td>
<td>$O(\log^6 \log n)$</td>
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</table>
## Lower Bounds

<table>
<thead>
<tr>
<th>Deterministic and Randomized</th>
<th>MIS and Maximal Matching</th>
<th>$(\Delta+1)$-coloring and $(2\Delta-1)$-edge-coloring</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>$\Omega\left(\min{ \log \Delta / \log\log \Delta, \sqrt{\log n / \log\log n} \right)$</td>
<td>$\Omega(\log^* n)$</td>
</tr>
</tbody>
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Kuhn, Moscibroda, Wattenhofer (2004)

Linial (1987)
Naor (1990)
Randomized Algorithms
Randomized algorithm for \((\Delta+1)\)-coloring

**Algorithm** (Barenboim and Elkin, 2013) for node \(u\)

\[
\textbf{while} \text{ uncolored do} \\
\quad \mathcal{C} = \{\text{colors previously adopted by neighbors}\} \\
\quad \text{pick } \ell(u) \text{ at random in } \{0, 1, \ldots, \Delta+1\} - \mathcal{C} \\
\quad \quad \text{• } 0 \text{ is picked w/ probability } \frac{1}{2} \\
\quad \quad \text{• } \ell(u) \in \{1, \ldots, \Delta+1\} - \mathcal{C} \text{ is picked w/ proba } \frac{1}{2(\Delta+1-|\mathcal{C}|)} \\
\quad \text{if } \ell(u) \neq 0 \text{ and } \ell(u) \notin \{\text{colors picked by neighbors}\} \\
\quad \quad \text{then adopt } \ell(u) \text{ as my color} \\
\quad \quad \text{else remain uncolored} \\
\quad \text{inform neighbors of status} \\
\]

1 round

1 round
**Definition** A sequence \((\mathcal{E}_n)_{n \geq 1}\) of events holds with high probability (whp) whenever \(\Pr[\mathcal{E}_n] = 1 - O(1/n^c)\) for some constant \(c > 0\).

**Theorem** (Barenboim and Elkin, 2013) The \((\Delta+1)\)-coloring algorithm takes, w.h.p., \(O(\log n)\) rounds.

Recall:

- \(\Pr[A|B] = \frac{\Pr[A \land B]}{\Pr[B]}\)
- \(\Pr[A] = \Pr[A|B] \cdot \Pr[B] + \Pr[A|\neg B] \cdot \Pr[\neg B]\)
- Union bound: \(\Pr[A \lor B] \leq \Pr[A] + \Pr[B]\)

\[\Pr[\exists s \in S : s \models \mathcal{P}] = \Pr[(s_1 \models \mathcal{P}) \lor (s_2 \models \mathcal{P}) \lor \ldots \lor (s_m \models \mathcal{P})]\]
Claim  For every node $u$, at any round, $\Pr[u \text{ terminates}] \geq \frac{1}{4}$

$$
\Pr[u \text{ termine}] = \Pr[\ell(u) \neq 0 \text{ et aucun } v \in N(u) \text{ satisfait } \ell(v) = \ell(u)]
= \Pr[\forall v \in N(u), \ell(v) \neq \ell(u) | \ell(u) \neq 0] \cdot \Pr[\ell(u) \neq 0]
= \frac{1}{2} \cdot \Pr[\forall v \in N(u), \ell(v) \neq \ell(u) | \ell(u) \neq 0]
$$

$$
\Pr[\ell(v) = \ell(u) | \ell(u) \neq 0] = \Pr[\ell(v) = \ell(u) | \ell(u) \neq 0 \land \ell(v) = 0] \Pr[\ell(v) = 0]
+ \Pr[\ell(v) = \ell(u) | \ell(u) \neq 0 \land \ell(v) \neq 0] \Pr[\ell(v) \neq 0]
= \Pr[\ell(v) = \ell(u) | \ell(u) \neq 0 \land \ell(v) \neq 0] \Pr[\ell(v) \neq 0]
\leq \frac{1}{2} \Pr[\ell(v) = \ell(u) | \ell(u) \neq 0 \land \ell(v) \neq 0]
= \frac{1}{2} \frac{1}{\Delta + 1 - |C(u)|}.
$$

$$
\Pr[\exists v \in N(u): \ell(v) = \ell(u) | \ell(u) \neq 0] \leq (\Delta - |C(u)|) \frac{1}{2(\Delta + 1 - |C(u)|)} < \frac{1}{2}
$$
\[ O(\log n) \text{ rounds w.h.p.} \]

\[ \Pr[u \text{ does not terminate in } k \ln(n) \text{ rounds}] \leq \left( \frac{3}{4} \right)^k n \ln \left( \frac{n^{1/3}}{4} \right) \]

\[ \Pr[\exists u \text{ that does not terminate in } k \ln(n) \text{ rounds}] \leq n^{1-k \ln \left( \frac{4}{3} \right)} \]

Let \( c > 1 \), by choosing \( k = \frac{1+c}{\ln \left( \frac{4}{3} \right)} \), we get:

\[ \Pr[\text{all nodes terminates after } (1+c)/\ln \left( \frac{4}{3} \right) \ln(n) \text{ rounds}] \geq 1-1/n^c. \]
Randomized algorithm for MIS

**Algorithm** (Luby, 1986)

\[
mis(u) \in \{-1, 0, 1\} = \{\text{undecided, not in MIS, in MIS}\}
\]

At any given round: \( H = G[\{u : mis(u)=-1\}] \)

**Trick:** enforcing an order between nodes:

\( v \preceq u \iff deg_H(v) > deg_H(u) \)

or \((deg_H(v) = deg_H(u) \text{ and } ID(v) > ID(u))\)
Luby’s algorithm

One phase of the algorithm for node $u$ with $\text{mis}(u) = -1$

if $\deg_H(u) = 0$ then $\text{mis}(u) \leftarrow 1$
else join($u$) $\leftarrow$ true with proba $1/(2 \deg_H(u))$, false otherwise
exchange join with every $v \in N(u)$
free($u$) $\leftarrow$ $\nexists v \in N(u)$ such that $v \not\succ u$ and join($v$)=true
if (join($u$) = true and free($u$) = true) then $\text{mis}(u) \leftarrow 1$
exchange mis with every $v \in N(u)$
if (mis($u$) = -1 and $\exists v \in N(u)$ mis($v$)=1) then $\text{mis}(u) \leftarrow 0$
exchange mis with every $v \in N(u)$
Luby’s algorithm terminates in O(log n) rounds, w.h.p.

Structure of the proof:

1. \( \Pr[\text{mis}(u) = 1] \geq 1/(4 \deg_H(u)) \)

2. For a set \( \mathcal{N} \) of nodes,
   
   \[ u \in \mathcal{N} \Rightarrow \Pr[u \text{ terminates}] \geq 1/36 \]

3. For a large set \( \mathcal{E} \) of edges,
   
   \[ e \in \mathcal{E} \Rightarrow \Pr[e \text{ removed from } H] \geq 1/36 \]

4. Use concentration result (Chernoff bound) to get w.h.p.
Step 1

\[
\Pr[mis(u) \neq 1 \mid join(u)] = \Pr[\exists v \in N(u) : v \succ u \land join(v) \mid join(u)] \\
= \Pr[\exists v \in N(u) : v \succ u \land join(v)] \\
\leq \sum_{v \in N(u) : v \succ u} \Pr[join(v)] \\
= \sum_{v \in N(u) : v \succ u} \frac{1}{2 \deg(v)} \\
\leq \frac{\deg(u)}{2 \deg(u)} \\
\leq \frac{1}{2}
\]

\[
\Pr[mis(u) = 1] = \Pr[mis(u) = 1 \mid join(u)] \cdot \Pr[join(u)]
\]

\[
\Pr[mis(u) = 1] \geq \frac{1}{2} \cdot \frac{1}{2 \deg(u)} = \frac{1}{4 \deg(u)}.
\]
Step 2

A node \( u \) is large if
\[
\sum_{v \in N(u)} \frac{1}{2 \deg(v)} \geq \frac{1}{6}
\]

Claim: \( u \) large \( \Rightarrow \Pr[u \text{ terminates}] \geq 1/36 \)

- True if \( \exists v \in N(u) : \deg_H(v) \leq 2 \)
- \( \forall v \in N(u) \), if \( \deg_H(v) \geq 3 \) then
\[
\frac{1}{2 \deg(v)} \leq \frac{1}{6}
\]

\[\iff \exists S \subseteq N(u) : \frac{1}{6} \leq \sum_{v \in S} \frac{1}{2 \deg(v)} \leq \frac{1}{3}\]

\[
\Pr[E_1 \lor E_2 \lor \cdots \lor E_r] = \sum_i \Pr[E_i] - \sum_{i \neq j} \Pr[E_i \land E_j] + \sum_{i \neq j \neq k} \Pr[E_i \land E_j \land E_k] - \cdots
\]

\[\cdots + (-1)^{r+1} \Pr[E_1 \land \cdots \land E_r].\]
\[
\Pr[\text{mis}(u) \neq -1] \geq \Pr[\exists v \in S : \text{mis}(v) = 1] \\
\geq \sum_{v \in S} \Pr[\text{mis}(v) = 1] - \sum_{v,w \in S, v \neq w} \Pr[\text{mis}(v) = 1 \land \text{mis}(w) = 1].
\]

\[\implies \Pr[\text{mis}(u) \neq -1] \geq \sum_{v \in S} \Pr[\text{mis}(v) = 1] - \sum_{v,w \in S, v \neq w} \Pr[\text{join}(v) \land \text{join}(w)]\]

\[\geq \sum_{v \in S} \frac{1}{4 \deg(v)} - \sum_{v \in S} \sum_{w \in S} \frac{1}{2 \deg(v)} \cdot \frac{1}{2 \deg(w)}\]

\[\geq \left( \sum_{v \in S} \frac{1}{2 \deg(v)} \right) \left( \frac{1}{2} - \sum_{w \in S} \frac{1}{2 \deg(w)} \right)\]

\[\geq \frac{1}{6} \left( \frac{1}{2} - \frac{1}{3} \right) = \frac{1}{36}.\]
Step 3

An edge \( e = \{u, v\} \) is large if \( u \) or \( v \) is large

For \( e = \{u, v\} \) with \( u < v \), orient the edge \( u \to v \)

Claim For every small node \( u \), \( \deg^+(u) \geq 2 \deg^-(u) \)

Indeed: \( \deg^+(u) < 2 \deg^-(u) \iff \deg(u) < 3 \deg^-(u) \)

\[
S = \{ v \in N(u) : \deg(v) \leq \deg(u) \}
\]

\[
|S| \geq \deg^-(u) \implies |S| \geq |N(u)|/3
\]

\[
\sum_{v \in N(u)} \frac{1}{2 \deg(v)} \geq \sum_{v \in S} \frac{1}{2 \deg(v)} \geq \sum_{v \in S} \frac{1}{2 \deg(u)} \geq \frac{\deg(u)}{3} \cdot \frac{1}{2 \deg(u)} = \frac{1}{6}
\]

\[\square\]
Let \( m = |E(H)| \)

We have:

\[
\sum_{u \text{ petit}} \deg^-(u) \leq \frac{1}{2} \sum_{u \text{ petit}} \deg^+(u) \leq \frac{m}{2}
\]

\[
\Rightarrow \sum_{u \text{ grand}} \deg^-(u) \geq \frac{m}{2} \Rightarrow \text{at least } m/2 \text{ large edges}
\]

\( X_e \) = Bernoulli variable equal to 1 if \( e \) is removed from \( H \)

For \( e \) large, \( \Pr[X_e=1] \geq 1/36 \Rightarrow \mathbb{E}X_e \geq 1/36 \)

\( X = \sum_{e \text{ large}} X_e \Rightarrow \mathbb{E}X = \sum_{e \text{ large}} \mathbb{E}X_e \geq m/72 \)

Let \( p = \Pr[X \leq \frac{1}{2} \mathbb{E}X] \)

\[
\mathbb{E}X = \sum_{x=0}^{m} x \Pr[X = x] = \sum_{x=0}^{\frac{1}{2}\mathbb{E}X} x \Pr[X = x] + \sum_{x=\frac{1}{2}\mathbb{E}X+1}^{m} x \Pr[X = x] \leq \frac{1}{2} p \mathbb{E}X + (1 - p)m
\]

\[
\Rightarrow p \leq \frac{m - \mathbb{E}X}{m - \frac{1}{2} \mathbb{E}X} \leq \frac{m - \frac{1}{2} \mathbb{E}X}{m} \leq 1 - \frac{1}{144}.
\]

Let \( \mathcal{E} = \langle \text{at least } m/144 \text{ edges are removed from } H \rangle \)

\( \Pr[\mathcal{E}] \geq 1/144 \)
Step 4
Let $Y_1, Y_2, \ldots, Y_k$ be Bernouilli variables w/ parameter $q = 1/144$
Let $Y = Y_1 + Y_2 + \ldots + Y_k$

Remark: Let $\alpha = 144/143$. If $Y \geq \log_{\alpha} |E(G)|$ then termination.

Chernoff Inequality: $\forall \delta \in ]0,1[\, , \Pr[Y \leq (1 - \delta)\mathbb{E}Y] \leq e^{-\frac{1}{2} \delta^2 \mathbb{E}Y}$

We have $\mathbb{E}Y = kq$, so, with $\delta = \frac{1}{2}$, we get $\Pr[Y \leq \frac{kq}{2}] \leq e^{-\frac{kq}{8}}$

For $k = c \log_{\alpha} n$, we get $\Pr[Y \leq \frac{cq \log_{\alpha} n}{2}] \leq e^{-\frac{cq \log_{\alpha} n}{8}}$

Let $c = 4/q \implies \frac{1}{2} c q \log_{\alpha} n \geq \log_{\alpha} |E(G)|$ and $cq \geq 8 \ln(\alpha)$.

$\implies e^{-\frac{cq \log_{\alpha} n}{8}} = \frac{1}{n^{\frac{cq}{8 \ln \alpha}}} \leq \frac{1}{n}$. $\implies \Pr[Y \leq \log_{\alpha} m] \leq \frac{1}{n}$.

Thus Luby's algorithm terminates in $O(\log n)$ rounds w.h.p.
Deterministic ⇔ Randomized
Network Decomposition

**Definition** A \((d,c)\)-decomposition of an \(n\)-node graph \(G = (V, E)\) is a partition of \(V\) into clusters such that each cluster has diameter at most \(d\) and the cluster graph is properly colored with colors \(1, \ldots, c\).

**Theorem** [Linial and Saks (1993)]
Every graph has a \((O(\log n),O(\log n))\)-decomposition, and such a decomposition can be computed by a randomized algorithm in \(O(\log^2 n)\) rounds in the LOCAL model.

**Theorem** [Panconesi and Srinivasan (1992)]
A \((2^{O(\sqrt{\log n})},2^{O(\sqrt{\log n})})\)-decomposition can be computed deterministically in \(2^{O(\sqrt{\log n})}\) rounds in the LOCAL model.
Impact on coloring and MIS

**Lemma** Given a \((d,c)\)-decomposition, \((\Delta+1)\)-coloring and MIS can be solved in \(O(cd)\) rounds in the LOCAL model.

**Proof**

Proceed in \(c\) phases, each of \(O(d)\) rounds
Theorem [V. Rozhon and M. Ghaffari (2019)]
A (O(log n),O(log n))-decomposition can be computed deterministically in O(log^{O(1)} n) rounds in the LOCAL model.

Corollary (Δ+1)-coloring and MIS can be deterministically solved in O(log^{O(1)} n) rounds in the LOCAL model.
SLOCAL Model
M. Ghaffari, F Kuhn, Y. Maus (2017)

- Sequential variant of the LOCAL model:
  - nodes are considered sequentially, one by one
  - the current node computes its output based solely on the states of the nodes in the ball of radius $t$ around it

- $\text{LOCAL}(t) = \{\text{problems solvable in } t \text{ rounds}\}$
- $\text{SLOCAL}(t) = \{\text{problem solvable with balls of radius } t\}$
- $\text{P-LOCAL} = \text{LOCAL}(\log^{O(1)} n)$
- $\text{P-SLOCAL} = \text{SLOCAL}(\log^{O(1)} n)$
Completeness Results

In the LOCAL model, a problem $Q$ is $t$-reducible to another problem $P$ if

$$t$$-round algorithm for $P \Rightarrow t$$-round algorithm for $Q$.

$P$ is P-SLOCAL-complete if $P \in \text{P-SLOCAL}$, and any $Q \in \text{P-SLOCAL}$ is $O(\log^{O(1)}n)$-reducible to $P$.

**Theorem** [M. Ghaffari, F Kuhn, Y. Maus (2017)]
Computing a $(O(\log^{O(1)}n),O(\log^{O(1)}n))$-decomposition is P-SLOCAL-complete.

**Corollary** $\text{P-LOCAL} = \text{P-SLOCAL}$. 

Derandomization

For Locally Checkable Labeling (LCL) problems:

**Theorem** [M. Naor and L. Stockmeyer (1992)]
\[ \text{LOCAL}(O(1)) = \text{RLOCAL}(O(1)) \]

**Theorem** [L. Feuilloley and P. F. (2015)]
\[ \text{LOCAL}(O(1)) = \text{RLOCAL}(O(1)) \] also for randomly locally checkable problems.

**Theorem** [V. Rozhon and M. Ghaffari (2019)]
\[ \text{P-LOCAL} = \text{P-RLOCAL}. \]
Randomized Algorithms using Shattering

Pick ● or ○ u.a.r.

W.h.p., max length monochromatic interval ≤ O(log n)

3-coloring or MIS: #rounds ≈ \text{Det}(O(\log n))
Graph Shattering

1. Shatter the graph using randomization
2. Complete each piece deterministically

parts that are fixed after 1.

parts that remain to be fixed by 2.

$\operatorname{Rand}(n) \approx \operatorname{Det}(O(\log^{O(1)} n))$
Deterministic lower bounds

Randomized lower bounds

**Theorem** [Y.-J. Chang, T. Kopelowitz, S. Pettie (2016)]
For any LCL problem in the LOCAL model, its randomized complexity on instances of size $n$ is at least its deterministic complexity on instances of size $\sqrt{\log n}$.

**Conclusion**: one needs to design better deterministic algorithms for improving the performances of randomized algorithms!
Concluding remarks
## Round Complexity

<table>
<thead>
<tr>
<th></th>
<th>MIS</th>
<th>(Δ+1)-coloring</th>
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</thead>
<tbody>
<tr>
<td><strong>Deterministic</strong></td>
<td>(O(\log^{O(1)}n))</td>
<td>(O(\log^{O(1)}n))</td>
</tr>
<tr>
<td></td>
<td>(\text{Rozhon, Ghaffari (2019)})</td>
<td>(\text{Rozhon, Ghaffari (2019)})</td>
</tr>
<tr>
<td><strong>Randomized</strong></td>
<td>(O(\log^{O(1)}\log n)+O(\log \Delta))</td>
<td>(O(\log^{O(1)}\log n))</td>
</tr>
<tr>
<td><strong>Maximal Matching</strong></td>
<td></td>
<td>(\text{(2Δ-1)-edge-coloring})</td>
</tr>
<tr>
<td><strong>Deterministic</strong></td>
<td>(O(\log^3n))</td>
<td>(O(\log^6n))</td>
</tr>
<tr>
<td></td>
<td>(\text{Fisher (2017)})</td>
<td>(\text{Ghaffari, Fisher, Kuhn (2017)\text{Ghaffari, Harris, Kuhn (2018)}})</td>
</tr>
<tr>
<td><strong>Randomized</strong></td>
<td>(O(\log^3\log n)+O(\log \Delta))</td>
<td>(O(\log^6\log n))</td>
</tr>
<tr>
<td></td>
<td>(\text{Barenboim, Elkin, Pettie, Schneider (2012)})</td>
<td>(\text{Elkin, Pettie, Su (2015)})</td>
</tr>
</tbody>
</table>
## Lower Bounds

<table>
<thead>
<tr>
<th>Deterministic and Randomized</th>
<th>MIS and Maximal Matching</th>
<th>$(\Delta+1)$-coloring and $(2\Delta-1)$-edge-coloring</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega\left(\min\left{ \log \Delta / \log \log \Delta, \sqrt{\log n / \log \log n} \right}\right)$</td>
<td>$\Omega(\log^* n)$</td>
<td></td>
</tr>
</tbody>
</table>

Kuhn, Moscibroda, Wattenhofer (2004)

Linial (1987)
Naor (1990)
Open problems

- Improve the constants (i.e., the degrees of the polylog)
- Close the gaps between lower and upper bounds
- Is $(\Delta+1)$-coloring solvable in $O(\log^*n)$ rounds?