Solution for the Exercise on Page 5 (Thursday Session)

Exercise: Show that

$$
\begin{aligned}
R_C^{k-means}(w) &:= \sum_{i \in C} w_i \|x_i - z_C(w)\|^2 \\
&\stackrel{!}{=} \frac{1}{2S_C(w)} \cdot \sum_{i,j \in C} w_i w_j \|x_i - x_j\|^2 \quad .
\end{aligned}
$$

Solution: For sake of brevity, let $W := S_C(w)$. Then:

$$
\begin{aligned}
R_C^{k-means}(w) &= \sum_{i \in C} w_i \left\langle x_i - \frac{1}{W} \sum_{j \in C} w_j x_j, x_i - \frac{1}{W} \sum_{k \in C} w_k x_k \right\rangle \\
&= \sum_{i \in C} w_i \left(\langle x_i, x_i \rangle + \frac{1}{W^2} \sum_{j,k \in C} w_j w_k \langle x_j, x_k \rangle \right. \\
&\quad \left. - \frac{2}{W} \sum_{j \in C} w_j \langle x_i, x_j \rangle \right) \\
&= \sum_{i \in C} \left(w_i \langle x_i, x_i \rangle + \frac{1}{W} w^\top X w - \frac{2}{W} w^\top X w \right) \\
&= \sum_{i \in C} w_i \langle x_i, x_i \rangle - \frac{1}{W} w^\top X w \quad ,
\end{aligned}
$$

where X is the matrix with entries $\langle x_i, x_j \rangle$. On the other hand:

$$
\begin{aligned}
\frac{1}{2W} \cdot \sum_{i,j \in C} w_i w_j \|x_i - x_j\|^2 &= \frac{1}{2W} \sum_{i,j \in C} w_i w_j \langle x_i - x_j, x_i - x_j \rangle \\
&= \frac{1}{2W} \sum_{i,j \in C} w_i w_j \left(\langle x_i, x_i \rangle + \langle x_j, x_j \rangle - 2 \langle x_i, x_j \rangle \right) \\
&= \sum_{i \in C} w_i \langle x_i, x_i \rangle - \frac{1}{W} w^\top X w \quad .
\end{aligned}
$$