Solution for Exercise 6 (Wednesday Session)

Lemma 0.1 *Assume that $\mathcal{U}$ is a linear subspace of $\mathbb{R}^n$ of dimension $n-1$, $\mu \notin \mathcal{U}$, and f is homogeneous (say of degree α). Then, $k(\mathcal{U}) = k(\mathbb{R}^n)$. Moreover, for $k := k(\mathcal{U}) = k(\mathbb{R}^n)$, T_k is positive semidefinite (or negative semidefinite, indefinite, respectively) on $\mathcal{U}$ if and only T_k is positive semidefinite (or negative semidefinite, indefinite, respectively) on $\mathbb{R}^n$.*

Proof $T_0 = T_0(h)$ is the constant function with value $f(\mu)$. If $f(\mu) \neq 0$, then $k(\mathcal{U}) = k(\mathbb{R}^n) = 0$. We can therefore assume that $f(\mu) = 0$ (which implies that $k(\mathbb{R}^n) \geq 1$).
Let $k := k(\mathcal{U})$. Clearly, $k(\mathbb{R}^n) \leq k$. In order to prove that $k(\mathbb{R}^n) \geq k$, it suffices to show that $T_0, \ldots, T_{k-1}$ vanish on $\mathbb{R}^n$. For $T_0 = f(\mu) = 0$, this holds. Assume inductively that, for some $l \in \{1, \ldots, k-1\}$, T_{l-1} vanishes on $\mathbb{R}^n$. This implies that $(\nabla^{l-1} f(\mu))_{i_1,\ldots,i_{l-1}} = 0$ for every sequence $1 \leq i_1, \ldots, i_{l-1} \leq n$. Since $\mathcal{U}$ is $(n-1)$-dimensional and $\mu \notin \mathcal{U}$, an arbitrary vector from $\mathbb{R}^n$ can be written in the form $u + \xi$ where $u \in \mathcal{U}$ and $\xi = \gamma\mu$ is a scalar multiple of μ. We proceed with a calculation which demonstrates that $T_l(u + \xi) = 0$:

$$\begin{aligned}
l! T_l(u+\xi) &= \sum_{i_1,\ldots,i_l} (\nabla^l f(\mu))_{i_1,\ldots,i_l} (u_{i_1} + \xi_{i_1}) \cdots (u_{i_l} + \xi_{i_l}) \\
&\stackrel{*}{=} l! T_l(u) + \sum_{i_1,\ldots,i_{l-1}} (u_{i_1} + \xi_{i_1}) \cdots (u_{i_{l-1}} + \xi_{i_{l-1}}) \sum_{i_l} (\nabla^l f(\mu))_{i_1,\ldots,i_l} \xi_{i_l} \\
&\quad + \sum_{i_1,\ldots,i_{l-2},i_l} (u_{i_1} + \xi_{i_1}) \cdots (u_{i_{l-2}} + \xi_{i_{l-2}}) u_{i_l} \sum_{i_{l-1}} (\nabla^l f(\mu))_{i_1,\ldots,i_l} \xi_{i_{l-1}} \\
&\quad + \cdots + \sum_{i_2,\ldots,i_l} u_{i_2} \cdots u_{i_l} \sum_{i_1} (\nabla^l f(\mu))_{i_1,\ldots,i_l} \xi_{i_1} \\
&= l! T_l(u) + \gamma \cdot \sum_{i_1,\ldots,i_{l-1}} (u_{i_1} + \xi_{i_1}) \cdots (u_{i_{l-1}} + \xi_{i_{l-1}}) (\nabla (\nabla^{l-1} f(\mu))_{i_1,\ldots,i_{l-1}})^\top \mu \\
&\quad + \gamma \cdot \sum_{i_1,\ldots,i_{l-2},i_l} (u_{i_1} + \xi_{i_1}) \cdots (u_{i_{l-2}} + \xi_{i_{l-2}}) u_{i_l} (\nabla (\nabla^{l-1} f(\mu))_{i_1,\ldots,i_{l-2},i_l})^\top \mu \\
&\quad + \cdots + \gamma \cdot \sum_{i_2,\ldots,i_l} u_{i_2} \cdots u_{i_l} (\nabla (\nabla^{l-1} f(\mu))_{i_2,\ldots,i_l})^\top \mu \\
&= l! T_l(u) = 0
\end{aligned}$$

Here, all indices $i_1, \ldots, i_l$ range from 1 to n. In the equation marked "$*$", we simply applied the distributive law to the product $(u_{i_1} + \xi_{i_1}) \cdots (u_{i_l} + \xi_{i_l})$. In the final equations (claiming that the whole sum collapses to zero), we made use of the following facts:

- $T_l(u) = 0$ because T_l vanishes on $\mathcal{U}$ and $u \in \mathcal{U}$.
- According to Exercise 5, $(\nabla^{l-1} f(w))_{i_1,\ldots,i_{l-1}}$ is homogeneous of degree $\alpha - l + 1$. Euler's homogeneity relation (with w set to μ) then yields

  $$(\nabla (\nabla^{l-1} f(\mu))_{i_1,\ldots,i_{l-1}})^\top \mu = (\alpha - l + 1)(\nabla^{l-1} f(\mu))_{i_1,\ldots,i_{l-1}} = 0 \ ,$$

  where the last equation holds because, by our induction hypothesis, T_{l-1} vanishes on $\mathbb{R}^n$.

These observations complete the inductive proof for $k := k(\mathcal{U}) = k(\mathbb{R}^n)$.
In order to complete the proof, let us inspect the initial part of the above calculation which showed

that $T_l(u+\xi) = T_l(u)$. This part is not only valid for $l = 1, \dots, k-1$ but also for $l = k$. Thus, $T_k(u+\xi) = T_k(u)$. Therefore, positive semi-definiteness of T_k on $\mathcal{U}$ implies positive semi-definiteness of T_k on $\mathbb{R}^n$. Analogously for negative semi-definiteness. Clearly, indefiniteness of T_k on $\mathcal{U}$ implies indefiniteness of T_k on $\mathbb{R}^n$. •