Algebraic approach to exact algorithms,
Part IV: Matching connectivity matrix

Łukasz Kowalik

University of Warsaw

ADFOCS, Saarbrücken, August 2013
Matching connectivity matrix

Let $n$ be an even integer.

- $\mathcal{H}_n$ is a square matrix over the $\mathbb{Z}_2$ field with rows and columns labeled by all perfect matchings in $K_n$.
- $(\mathcal{H}_n)_{M_1,M_2} = [M_1 \cup M_2$ forms a Hamiltonian cycle in $K_t]$
- Dimension:
  \[
  \frac{n!}{(n/2)!2^{n/2}} = \frac{(\frac{n}{2})^n}{(\frac{n}{2e})^{n/2}2^{n/2}}n^{O(1)} = (\frac{n}{e})^{n/2}n^{O(1)} = 2^{O(n \log n)}
  \]

Example: $\mathcal{H}_4$

<table>
<thead>
<tr>
<th></th>
<th>$^\wedge\wedge$</th>
<th>$^\wedge\wedge\wedge$</th>
<th>$^\wedge\wedge\wedge\wedge$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^\wedge\wedge$</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$^\wedge\wedge\wedge$</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$^\wedge\wedge\wedge\wedge$</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
Matching connectivity matrix $\mathcal{H}_6$

<table>
<thead>
<tr>
<th>Nr.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>11</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>14</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>15</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Matching connectivity matrix $\mathcal{H}_n$

- $\mathcal{H}_n$ is huge

What is the rank of $\mathcal{H}_n$?
Matching connectivity matrix $\mathcal{H}_n$

- $\mathcal{H}_n$ is huge
- $\mathcal{H}_n$ has much redundancy
Matching connectivity matrix $\mathcal{H}_n$

- $\mathcal{H}_n$ is huge
- $\mathcal{H}_n$ has much redundancy
- What is the rank of $\mathcal{H}_n$?
Family of matchings $X_n$

Partition the vertices $1, 2, \ldots, n$ into $n/2 + 1$ groups:

$$1 \mid 23 \mid 45 \mid \cdots \mid (n-2)(n-1) \mid n$$

Let $\text{pm}(G)$ denote the set of all perfect matchings of $G$.

$$X_n = \{ M \in \text{pm}(K_n) : M \text{ matches vertices from neighboring groups only} \}$$

**Example: $n = 6$**

Groups:

$$1 \mid 23 \mid 45 \mid 6$$

Matchings:

$$X_6 = \{ \{12, 34, 56\}, \{12, 35, 46\}, \{13, 24, 56\}, \{13, 25, 46\} \}$$
Indexing the matchings from $X$

- $X$ has $2^{n/2-1}$ matchings.
- The matchings are indexed by 0/1-strings of length $n/2 - 1$.
- Building a matching from the string $w_1 \ldots, w_{n/2-1}$:
  - For $i = 1, \ldots, n/2 - 1$:
    - if $w_i = 1$ then the yet unmatched vertex of $i$-th group is matched with the first vertex of the $(i + 1)$-th group,
    - if $w_i = 0$ then ... with the second ...

Example: $n = 6$

Groups:

<table>
<thead>
<tr>
<th>Group 1</th>
<th>Group 2</th>
<th>Group 3</th>
<th>Group 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>23</td>
<td>45</td>
<td>6</td>
</tr>
</tbody>
</table>

Matchings:

- $X(11) = \{12, 34, 56\}$
- $X(10) = \{12, 35, 46\}$
- $X(01) = \{13, 24, 56\}$
- $X(00) = \{13, 25, 46\}$
For $w \in \{0, 1\}^\ell$ denote $\overline{w} = w \oplus 1 \cdots 1$, e.g. $\overline{110} = 001$. 
Properties of the $\mathcal{H}_{X,Y}$ submatrix

For $w \in \{0, 1\}^\ell$ denote $w = w \text{xor } 1 \cdots 1$, e.g. $110 = 001$.

**Observation**

$X(w) \cup X(u)$ is a Hamiltonian cycle iff $w = \overline{u}$.
Properties of the $H_{X,X}$ submatrix

For $w \in \{0, 1\}^\ell$ denote $\overline{w} = w \text{xor } 1 \cdots 1$, e.g. $\overline{110} = 001$.

Observation

$X(w) \cup X(u)$ is a Hamiltonian cycle iff $w = \overline{u}$.

Proof:
Assume $w_i = u_i$ for some $i$. 
Properties of the $\mathcal{H}_{x,x}$ submatrix

For $w \in \{0, 1\}^\ell$ denote $\overline{w} = w \text{xor} 1 \cdots 1$, e.g. $\overline{110} = 001$.

**Observation**

$X(w) \cup X(u)$ is a Hamiltonian cycle iff $w = \overline{u}$.

**Proof:**

Assume $w_i = u_i$ for some $i$.

$X(u) \cup X(w)$ has at least two connected components.
Properties of the $H_{x,x}$ submatrix

Observation

$X(w) \cup X(u)$ is a Hamiltonian cycle iff $w = \bar{u}$.

Proof:
Assume $w = \bar{u}$.
Properties of the $\mathcal{H}_{X,X}$ submatrix

Observation

$X(w) \cup X(u)$ is a Hamiltonian cycle iff $w = \overline{u}$.

Proof:
Assume $w = \overline{u}$. Every group looks like:

- Every vertex is adjacent to a vertex in the previous group.

Every vertex is adjacent to a vertex in the previous group.
Properties of the $\mathcal{H}_{X,X}$ submatrix

Observation

$X(w) \cup X(u)$ is a Hamiltonian cycle iff $w = \bar{u}$.

Proof:
Assume $w = \bar{u}$. Every group looks like:

- Every vertex is adjacent to a vertex in the previous group.
- Hence, every vertex has a path to vertex 1.
Properties of the $H_{x,x}$ submatrix

Observation

$X(w) \cup X(u)$ is a Hamiltonian cycle iff $w = \overline{u}$.

Proof:
Assume $w = \overline{u}$. Every group looks like:

- Every vertex is adjacent to a vertex in the previous group.
- Hence, every vertex has a path to vertex 1.
- Hence there is only one connected component.
- Since all degrees are 2, this is a HC.
Properties of the $\mathcal{H}_{x,x}$ submatrix

Order the rows/columns of $\mathcal{H}_{x,x}$ in lexicographic order, i.e.:

$$x(0\cdots000), x(0\cdots001), x(0\cdots010), x(0\cdots011), \ldots, x(1\cdots111).$$

Then, $\mathcal{H}_{x,x} = \begin{bmatrix}
0 & \cdots & 0 & 1 \\
0 & \cdots & 1 & 0 \\
\vdots & \vdots & 0 & 0 \\
1 & \cdots & 0 & 0
\end{bmatrix}$, so rank $\mathcal{H}_{x,x} = 2^{n/2} - 1$.
Properties of the $\mathcal{H}_{\mathbf{X},\mathbf{X}}$ submatrix

Order the rows/columns of $\mathcal{H}_{\mathbf{X},\mathbf{X}}$ in lexicographic order, i.e.:

\[
\mathbf{X}(0 \cdots 000), \mathbf{X}(0 \cdots 001), \mathbf{X}(0 \cdots 010), \mathbf{X}(0 \cdots 011), \ldots, \mathbf{X}(1 \cdots 111).
\]

Then, $\mathcal{H}_{\mathbf{X},\mathbf{X}} = \begin{bmatrix}
0 & \cdots & 0 & 1 \\
0 & \cdots & 1 & 0 \\
\vdots & \ddots & 0 & 0 \\
1 & \cdots & 0 & 0
\end{bmatrix}$, so $\text{rank } \mathcal{H}_{\mathbf{X},\mathbf{X}} = 2^{n/2-1}$.

Corollary

$\text{rank } \mathcal{H}_{n} \geq 2^{n/2-1}$. 
Properties of the $H_{x,x}$ submatrix

Order the rows/columns of $H_{x,x}$ in lexicographic order, i.e.:

$X(0 \cdots 000), X(0 \cdots 001), X(0 \cdots 010), X(0 \cdots 011), \ldots, X(1 \cdots 111)$.

Then, $H_{x,x} =$

$$
\begin{bmatrix}
0 & \ldots & 0 & 1 \\
0 & \ldots & 1 & 0 \\
\vdots & \ddots & 0 & 0 \\
1 & \ldots & 0 & 0
\end{bmatrix},
$$
so rank $H_{x,x} = 2^{n/2-1}$.

Corollary

rank $H_n \geq 2^{n/2-1}$.

Rows $X$ of $H$ are linearly independent.

Question

Do they form a basis of the row space of $H$?
Assume that $X$ is a basis of the row space of $\mathcal{H}$:
For any $M \in \text{pm}(K_n)$, for some $c_{M,w} \in \{0, 1\}$,

$$
\mathcal{H}_M = \sum_{w \in \{0,1\}^{n/2-1}} c_{M,w} \mathcal{H}_{X(w)}
$$

- $\mathcal{H}_{M,X(w)} = 0 \Rightarrow c_{M,w} = 0$,
- $\mathcal{H}_{M,X(w)} = 1 \Rightarrow c_{M,w} = 1$.

Hence, $c_{M,w} = \mathcal{H}_{M,X(w)}$ and

$$
\mathcal{H}_M = \sum_{w \in \{0,1\}^{n/2-1}} \mathcal{H}_{M,X(w)} \mathcal{H}_{X(w)}
$$
The representation formula

If $X$ is a basis then $H_M = \sum_{w \in \{0,1\}^{n/2-1}} H_{M,X(w)} H_{X(w)}$. Then,

The representation formula

$$H_{M_1,M_2} = \sum_{w \in \{0,1\}^{n/2-1}} H_{M_1,X(w)} H_{X(w),M_2}.$$ 

Theorem (Cygan, Kratsch, Nederlof 2013)

The representation formula holds.

(technical inductive proof skipped here.)

Corollary (Cygan, Kratsch, Nederlof 2013)

$$\text{rank } H_n = 2^{n/2} - 1.$$ 

Note: The representation formula holds in $GF(2^k)$ for every $k$. 
Let $G = (V, E)$ be an undirected graph.

We want to test Hamiltonicity of $G$.

W.l.o.g. $|V|$ is even.
Undirected Hamiltonicity in $O^*(1.888^n)$ time

- Let $G = (V, E)$ be an undirected graph.
- We want to test Hamiltonicity of $G$.
- W.l.o.g. $|V|$ is even.
- Yet another hero (over $GF(2^{2n})$):

$$P(x, y) = \sum_{M_1, M_2 \in pm(G)} \prod_{e \in M_1} x_e \prod_{e \in M_2} y_e$$

$M_1 \cup M_2$ is a HC
Polynomial $P$ and Hamiltonicity

$$P(x, y) = \sum_{M_1, M_2 \in \text{pm}(G)} \prod_{e \in M_1} x_e \prod_{e \in M_2} y_e$$

$M_1 \cup M_2$ is a HC

**Observation**

$P \neq 0$ iff $G$ is Hamiltonian.

**Proof:**

$(\Rightarrow)$: Obvious.

$(\Leftarrow)$: Let $H$ be a HC in $G$.

Then $H = M_1 \cup M_2$ where $M_1$, $M_2$ are perfect matchings,

The sum in the definition of $P$ contains each of the monomials $\prod_{e \in M_1} x_e \prod_{e \in M_2} y_e$ and $\prod_{e \in M_2} x_e \prod_{e \in M_1} y_e$ exactly once.
Rewriting $P$

\[
P(\{x_e\}_{e \in E}, \{y_e\}_{e \in E}) = \sum_{M_1, M_2 \in \text{pm}(G)} \prod_{e \in M_1} x_e \prod_{e \in M_2} y_e = \]

\[
\sum_{M_1 \in \text{pm}(G)} \sum_{M_2 \in \text{pm}(G)} H_{M_1, M_2} \prod_{e \in M_1} x_e \prod_{e \in M_2} y_e = (RF)
\]

\[
\sum_{M_1 \in \text{pm}(G)} \sum_{M_2 \in \text{pm}(G)} \sum_{w \in \{0, 1\}^{n/2-1}} H_{M_1, X(w)} H_{X(w), M_2} \prod_{e \in M_1} x_e \prod_{e \in M_2} y_e =
\]

\[
\sum_{w \in \{0, 1\}^{n/2-1}} \left( \sum_{M_1 \in \text{pm}(G)} H_{M_1, X(w)} \prod_{e \in M_1} x_e \right) \cdot \left( \sum_{M_2 \in \text{pm}(G)} H_{X(w), M_2} \prod_{e \in M_2} y_e \right)
\]

where for any $M \in X$,

\[
\text{ext}^G_M(\{z_e\}_{e \in E}) = \sum_{M' \in \text{pm}(G)} \prod_{e \in M'} z_e
\]

where $\text{ext}^G_M(\{z_e\}_{e \in E})$ is a HC, $M' \in \text{pm}(G)$, and $M \cup M'$ is a HC.
Evaluating $P$ in $O^*(1.888^n)$ time

We got:

$$P(\{x_e\}_{e \in E}, \{y_e\}_{e \in E}) = \sum_{w \in \{0,1\}^{n/2-1}} \text{ext}_{\bar{X}_w}^G(\{x_e\}_{e \in E}) \text{ext}_{\bar{X}_w}^G(\{y_e\}_{e \in E})$$

- Note that $|X| = 2^{n/2-1} = O(1.42^n)$.
- Hence it suffices to precompute $\text{ext}_M^G(\{x_e\}_{e \in E})$ and $\text{ext}_M^G(\{y_e\}_{e \in E})$ for all $M \in X$ in $O^*(1.888^n)$ time.
Evaluating \( \text{ext}_M^G(\{z_e\}_{e \in E}) = \sum_{M' \in \text{pm}(G)} \prod_{e \in M'} z_e \) in \( O^*(1.888^n) \).

Fix any \( u_0 \in V \).

**N-alternating \( v \)-walk**

Let \( N \) be a matching in \( K_n \).

A walk \( u_0, u_1, \ldots, u_t \) in \( K_n \) is called **N-alternating \( v \)-walk** if

- for every \( i = 0, \ldots, n/2 - 1 \), \( u_{2i}u_{2i+1} \in N \) and \( u_{2i+1}u_{2i+2} \in E(G) \).
- \( t = 2|N| \),
- each edge of \( N \) is visited,
- \( u_t = v \),

\[ u_0 u_1 \ldots u_{|N|} = v \]
Evaluating $\text{ext}_M^G\left(\{z_e\}_{e \in E}\right) = \sum \prod z_e$ in $O^*(1.888^n)$.

Fix any $u_0 \in V$.

**$N$-alternating $v$-walk**

For every matching $N$ such that $N \subseteq M'$ for some $M' \in X$, for every $v \in V$, compute

$$T[N, v] = \sum_{\text{$N$-alternating $v$-walk}} \prod_{e_1, e_2, \ldots, e_2|N|} z_{e_2i}$$
Evaluating $\text{ext}_M^G(\{z_e\}_{e \in E}) = \sum_{M' \in \text{pm}(G)} \prod_{e \in M'} z_e$ in $O^*(1.888^n)$. 

Fix any $u_0 \in V$.

**N-alternating v-walk**

For every matching $N$ such that $N \subseteq M'$ for some $M' \in X$, for every $v \in V$, compute

$$T[N, v] = \sum_{\text{N-alternating } v\text{-walk } i=1}^{\mid N \mid} \prod_{e_1, e_2, \ldots, e_2 \mid N} z_{e_2i}$$

Note that $\text{ext}_M^G(\{z_e\}_{e \in E}) = T[M, u_0]$. 
Evaluating $\operatorname{ext}_M^G(\{z_e\}_{e \in E}) = \sum_{M' \in \text{pm}(G)} \prod_{e \in M'} z_e$ in $O^*(1.888^n)$.

$M' \cup M'$ is a HC

Fix any $u_0 \in V$.

**N-alternating $v$-walk**

For every matching $N$ such that $N \subseteq M'$ for some $M' \in X$, for every $v \in V$, compute

$$T[N, v] = \sum_{\text{$N$-alternating $v$-walk} \atop e_1, e_2, \ldots, e_2|N|} \prod_{i=1}^{|N|} z_{e_2i}$$

Note that $\operatorname{ext}_M^G(\{z_e\}_{e \in E}) = T[M, u_0]$.

**Dynamic programming formula**

$$T[N, v] = \sum_{uv \in E} \sum_{u'u \in N} z_{uv} T[N \setminus \{u' u\}, u']$$
Evaluating $\text{ext}_M^G(\{z_e\}_{e \in E}) = \sum_{M' \in \text{pm}(G)} \prod_{e \in M'} z_e$ in $O^*(1.888^n)$.

**Dynamic programming formula**

$$T[N, v] = \sum_{uv \in E} \sum_{u' \in N} z_{uv} T[N \setminus \{u'u\}, u']$$

**Corollary**

Let $\alpha(n) = |\{N \subseteq M : M \in X_n\}|$.

All entries of $T[N, v]$ can be computed in $O^*(\alpha(n))$ time.

- Since $|X_n| = 2^{n/2-1}$ and every $M \in X_n$ has $2^{n/2}$ subsets, $\alpha(n) \leq 2^{n-1}$.
Evaluating \( \text{ext}_M^G(\{z_e\}_{e \in E}) = \sum_{M' \in \text{pm}(G)} \prod_{e \in M'} z_e \) in \( O^*(1.888^n) \).

\( M' \in \text{pm}(G) \) \( e \in M' \)
\( M \cup M' \) is a HC

Dynamic programming formula

\[
T[N, v] = \sum_{uv \in E} \sum_{u' \in N} z_{uv} T[N \setminus \{u' u\}, u']
\]

Corollary

Let \( \alpha(n) = |\{N \subseteq M : M \in \mathbf{X}_n\}|. \)

All entries of \( T[N, v] \) can be computed in \( O^*(\alpha(n)) \) time.

- Since \( |\mathbf{X}_n| = 2^{n/2-1} \) and every \( M \in \mathbf{X}_n \) has \( 2^{n/2} \) subsets, \( \alpha(n) \leq 2^{n-1} \).
- ... but there are a lot of common subsets!
Bounding $\alpha(n)$

Let $\beta(n) = |\{N \subseteq M : M \in X_n \text{ and } n \notin V(N)\}|$.

Then

\[
\begin{align*}
\alpha(n) &= \begin{cases} 
2\alpha(n - 2) + \beta(n) & \text{match vertex } n \\
4\alpha(n - 4) + 1 \cdot \beta(n - 2) & \text{do not match vertex } n
\end{cases} \\
\beta(n) &= \begin{cases} 
\alpha(n - 2) & \text{match } n - 2 \text{ or } n - 1 \\
\beta(n - 2) & \text{do not match them}
\end{cases}
\end{align*}
\]

Solve it using your favorite method and get

$\alpha(n) = O((\frac{3+\sqrt{17}}{2})^{n/2}) = O(1.88721^n)$.
Theorem (Cygan, Kratsch, Nederlof 2013)

The Hamiltonian cycle problem in undirected graphs can be solved in $O^*(1.888^n)$ time.
Hamiltonicity in directed bipartite graphs in $O^*(1.888^n)$ time

$G = (V_1 \cup V_2, E)$ – a directed bipartite graph.
Hamiltonicity in directed bipartite graphs in $O^*(1.888^n)$ time

$G = (V_1 \cup V_2, E)$ – a directed bipartite graph.

$E_1 = \{(v_1, v_2) \in E : v_1 \in V_1 \text{ and } v_2 \in V_2\}; \ G_1 = (V, E_1)$

$E_2 = \{(v_2, v_1) \in E : v_1 \in V_1 \text{ and } v_2 \in V_2\}; \ G_2 = (V, E_2)$

$P = \sum_{M_1 \in pm(G_1)} \prod_{e \in M_1} x_e \prod_{e \in M_2} y_e = \sum_{w \in \{0,1\}^{n/2-1}} \text{ext}_{X(w)}^{G_1}(\{x_e\}_{e \in E_1}) \text{ext}_{X(w)}^{G_2}(\{y_e\}_{e \in E_2})$

$M_1 \cup M_2$ is a HC
Theorem (Cygan, Kratsch, Nederlof 2013)

The Hamiltonian cycle problem in directed bipartite graphs can be solved in $O^*(1.888^n)$ time.