Algebraic approaches to exact algorithms, part V: Systems of linear equations

Łukasz Kowalik

University of Warsaw

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We find some related quantities \( x_2, \ldots, x_t \).
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There are $t$ linear equations in variables $x_1, \ldots, x_t$ such that:
- we can show the equations are linearly independent,
- we can compute the coefficients and the constant terms of the equations efficiently.
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There are $t$ linear equations in variables $x_1, \ldots, x_t$ such that:
- we can show the equations are linearly independent,
- we can compute the coefficients and the constant terms of the equations efficiently.

We solve the system using Gaussian Elimination in $O(t^3)$ time. (Or in $O(t^\omega)$ time if it matters.)
Recap: Fast Zeta transform $\zeta$

Let $f : 2^U \to \mathbb{N}$.

Zeta transform

$$(\zeta f)(X) = \sum_{Y \subseteq X} f(Y).$$

Trimmed zeta transform (Björklund, Husfeldt, Kaski, Koivisto)

- For any set family $\mathcal{G} \subseteq 2^U$ we can compute all values of $\zeta f|_\mathcal{G}$ in $O^*(|\downarrow\mathcal{G}|)$ time.
- We can compute all values of $\zeta f$ in $O^*(|\uparrow\text{supp}(f)|)$ time.
Recap: Fast up-zeta transform $\zeta^\uparrow$

Let $f : 2^U \to \mathbb{N}$.

**Up-zeta transform**

$$(\zeta^\uparrow f)(X) = \sum_{Y \supseteq X} f(Y).$$

**Trimmed up-zeta transform (Björklund, Husfeldt, Kaski, Koivisto)**

- For any set family $\mathcal{G} \subseteq 2^U$ we can compute all values of $\zeta^\uparrow f|_{\mathcal{G}}$ in $O^*(|\uparrow \mathcal{G}|)$ time.
- We can compute all values of $\zeta^\uparrow f$ in $O^*(|\downarrow \text{supp}(f)|)$ time.
Intersection Transform

• $U$ is a given set, $|U| = n$.
• We are given $\mathcal{F}, \mathcal{G} \subseteq 2^U$.
• For every $Y \in \mathcal{G}$ and every $\ell \in \{0, \ldots, n\}$, compute

\[ \iota_{\mathcal{F}}(\ell, Y) = |\{X \in \mathcal{F} : |X \cap Y| = \ell\}| \]

• $n + 1$ indeterminates $x^Y_\ell = \iota_{\mathcal{F}}(\ell, Y)$, for $\ell = 0, \ldots, n$
• $n + 1$ linear equations?
Intersection Transform: linear equations

For every \( Y \in G \) and \( \ell \in \{0, \ldots, n\} \), find \( x^Y_\ell \). \( x^Y_\ell = |\{X \in \mathcal{F} : |X \cap Y| = \ell\}|. \)

For every \( Y \in G \) and \( j \in \{0, \ldots, n\} \),

\[
b^Y_j = \sum_{Z \subseteq Y, |Z| = j} \left| \left\{ X \in \mathcal{F} : Z \subseteq X \right\} \right| =
\]

\[
= \sum_{Z \subseteq Y} \sum_{X \in \mathcal{F}, |Z| = j} 1 = \sum_{X \in \mathcal{F}} \sum_{Z \subseteq X} 1 = \sum_{X \in \mathcal{F}} \left( \begin{array}{c} |X \cap Y| \\ j \end{array} \right) =
\]

\[
= \sum_{\ell=0}^{n} \sum_{X \in \mathcal{F}, |X \cap Y| = \ell} \binom{\ell}{j} = \sum_{\ell=0}^{n} \binom{\ell}{j} \sum_{X \in \mathcal{F}, |X \cap Y| = \ell} 1 = \sum_{\ell=0}^{n} \binom{\ell}{j} x^Y_\ell
\]
Intersection Transform: linear equations

For every $Y \in G$ and $\ell \in \{0, \ldots, n\}$, find $x_\ell^Y = |\{X \in \mathcal{F} : |X \cap Y| = \ell\}|$.

For every $Y \in G$ we got $(n + 1)$ linear equations:

$$\sum_{\ell=0}^{n} \binom{j}{\ell} x_\ell^Y = b_j^Y, \quad j = 0, \ldots, n$$

where $b_j^Y = \sum_{\substack{Z \subseteq Y \mid |Z| = j}} |\{X \in \mathcal{F} : Z \subseteq X\}|$

- Since for $\ell < j$, $\binom{j}{\ell} = 0$, and $\binom{j}{\ell} = 1$ and the coefficients matrix is non-singular.
Intersection Transform: linear equations

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- Since for $\ell < j$, $\binom{j}{\ell} = 0$, and $\binom{j}{\ell} = 1$ and the coefficients matrix is non-singular.
- The coefficients can be evaluated fast.
Intersection Transform: linear equations

Intersection Transform

For every $Y \in G$ and $\ell \in \{0, \ldots, n\}$, find $x^Y_\ell = |\{X \in \mathcal{F} : |X \cap Y| = \ell\}|$.

For every $Y \in G$ we got $(n + 1)$ linear equations:

$$\sum_{\ell=0}^{n} \binom{j}{\ell} x^Y_\ell = b^Y_j, \quad j = 0, \ldots, n$$

where $b^Y_j = \sum_{\substack{Z \subseteq Y \mid |Z| = j}} |\{X \in \mathcal{F} : Z \subseteq X\}|$

- Since for $\ell < j$, $\binom{j}{\ell} = 0$, and $\binom{j}{\ell} = 1$ and the coefficients matrix is non-singular.
- The coefficients can be evaluated fast.
- How fast can we evaluate $b^Y_j$?
Evaluating $b_j^Y$ for every $Y \in \mathcal{G}$ and $j = 0, \ldots, n$

\[
b_j^Y = \sum_{Z \subseteq Y \atop |Z| = j} |\{X \in \mathcal{F} : Z \subseteq X\}| = \\
\sum_{Z \subseteq Y} \sum_{Z \supseteq X \atop |Z| = j} [X \in \mathcal{F}] = \sum_{Z \subseteq Y \atop |Z| = j} (\zeta^\uparrow \mathbf{1}_{\mathcal{F}})(Z) = (\zeta f)(Y),
\]

where for every $Z \in \downarrow \mathcal{G}$,

\[
f(Z) = (\zeta^\uparrow \mathbf{1}_{\mathcal{F}})(Z) \cdot [|Z| = j].
\]

Algorithm for evaluating $b_j^Y$ for every $Y \in \mathcal{G}$.

1. Compute $(\zeta^\uparrow \mathbf{1}_{\mathcal{F}})(Z)$ for all $Z \in \downarrow \mathcal{G}$ in $O(|\downarrow \text{supp}(\mathbf{1}_{\mathcal{F}})|) = O^*(|\downarrow \mathcal{F}|)$ time; from this compute $f(Z)$ easily.

2. Compute $(\zeta f)(Y)$ for all $Y \in \mathcal{G}$ in $O^*(\downarrow \mathcal{G})$ time.

Total running time: $O^*(|\downarrow \mathcal{F}| + |\downarrow \mathcal{G}|)$
Algorithm

1. Compute $b_j^Y$ for every $Y \in \mathcal{G}$ and $j = 0, \ldots, n$ in $O^*(|\downarrow \mathcal{F}| + |\downarrow \mathcal{G}|)$ time,

2. For every $Y \in \mathcal{G}$, solve the system of linear equations with indeterminates $x_\ell^Y$, $\ell = 0, \ldots, n$, using Gaussian Elimination in $O(n^3)$ time. (Actually one can derive an explicit formula, skipped here.)

Theorem (Björklund, Husfeldt, Kaski, Koivisto 2008)

Given $\mathcal{F}, \mathcal{G} \subseteq 2^U$, the values of

$$\nu_{\mathcal{F}}(Y, \ell) = |\{X \in \mathcal{F} : |X \cap Y| = \ell\}|$$

for all $Y \in \mathcal{G}$, $\ell = 0, \ldots, n$ can be found in $O^*(|\downarrow \mathcal{F}| + |\downarrow \mathcal{G}|)$ time.
With minor modifications to what we have just seen we can show:

**Theorem (Björklund, Husfeldt, Kaski, Koivisto 2008)**

Given $\mathcal{F}, \mathcal{G} \subseteq 2^U$, and a function $f : \mathcal{F} \to \mathbb{N}$, the values of

$$f_\ell(Y, \ell) = \sum_{X \in \mathcal{F} \mid |X \cap Y| = \ell} f(X)$$

for all $Y \in \mathcal{G}$, $\ell = 0, \ldots, n$ can be found in $O^*(|\downarrow \mathcal{F}| + |\downarrow \mathcal{G}|)$ time.

(By putting $f = 1_{\mathcal{F}}$ we get the previous version.)

**Corollary**

Given two functions $f, g : \binom{U}{q} \to \mathbb{N}$, we can compute the number

$$f \boxplus_\ell g = \sum_{X, Y \in \binom{U}{q} \mid |X \cap Y| = \ell} f(X)g(Y)$$

for all $\ell = 0, \ldots, n$ in $O^*(n^q)$ time.
With minor modifications to what we have just seen we can show:

**Theorem (Björklund, Husfeldt, Kaski, Koivisto 2008)**

Given $\mathcal{F}, \mathcal{G} \subseteq 2^U$, and a function $f : \mathcal{F} \rightarrow \mathbb{N}$, the values of

$$f_\ell(Y, \ell) = \sum_{X \in \mathcal{F}} f(X)$$

for $X \cap Y |X \cap Y| = \ell$ for all $Y \in \mathcal{G}$, $\ell = 0, \ldots, n$ can be found in $O^*(|\downarrow \mathcal{F}| + |\downarrow \mathcal{G}|)$ time.

(By putting $f = 1_\mathcal{F}$ we get the previous version.)

**Corollary**

Given two functions $f, g : \binom{U}{q} \rightarrow \mathbb{N}$, we can compute the number

$$f \boxdot_\ell g = \sum_{X, Y \in \binom{U}{q}} f(X)g(Y) = \sum_{Y \in \binom{U}{q}} g(Y) \sum_{X \in \binom{U}{q}} f(X)$$

for all $\ell = 0, \ldots, n$ in $O^*(n^q)$ time.
Theorem (Björklund, Husfeldt, Kaski, Koivisto 2008)

Given $\mathcal{F}, \mathcal{G} \subseteq 2^U$, the values of

$$\iota_\mathcal{F}(Y, \ell) = |\{X \in \mathcal{F} : |X \cap Y| = \ell\}|$$

for all $Y \in \mathcal{G}$, $\ell = 0, \ldots, n$ can be found in $O^*(|\downarrow \mathcal{F}| + |\downarrow \mathcal{G}|)$ time.

Corollary

Given two functions $f, g : \binom{U}{q} \to \mathbb{N}$, we can compute the number

$$f \mathcal{X}_\ell g = \sum_{X,Y \in \binom{U}{q}} f(X)g(Y)$$

for $X \cap Y = \ell$ for all $\ell = 0, \ldots, n$ in $O^*(n^q)$ time.
Application: counting $k$-paths in $O^*(n^{k/2})$ time

Problem

Given a directed graph $G = (V, E)$ count the number of $k$-vertex paths.

The problem is $\#W[1]$-hard when parameterized by $k$.

Algorithm (assume w.l.o.g. $k$ is even)

For every $v \in V$ find the number of paths where $v$ is the $k/2$-th vertex:

Define functions $f, g : \binom{V}{k/2} \to \mathbb{N}$

1. $f(S)$ is the number of paths $P$ that end in $v$ and $V(P) = S$;
2. $g(S)$ is the number of paths $P$ that start in $v$ and $V(P) = S \cup \{v\}$.

Compute $f \boxtimes_0 g = \sum_{X,Y \in \binom{V}{k/2}, |X \cap Y| = 0} f(X)g(Y)$ in $O^*(n^{k/2})$ time.
Counting $k$-paths by disjoint triples

Algorithm (w.l.o.g. assume $k$ is a multiple of 3)

For every $v_1, v_2 \in V(G)$ count paths where $v_i$ is the $\frac{i}{3} k$-th vertex:

Define functions $f, g, h : \binom{V}{k/3} \to \mathbb{N}$

1. $f(S)$ is the number of paths $P$ that end in $v_1$ and $V(P) = S$;
2. $g(S)$ is the number of paths $P$ from $v_1$ to $v_2$ and $V(P) = S \cup \{v_1\}$.
3. $h(S)$ is the number of paths $P$ that start in $v_2$ and $V(P) = S \cup \{v_2\}$.

Compute $\Delta(f, g, h) = \sum_{A, B, C \in \binom{U}{q}} f(A)g(B)h(C)$.

\[|A \cap B| = |A \cap C| = |B \cap C| = \emptyset\]
Problem (slightly simplified)

Let $q \leq |U|/3$. Given $\mathcal{F} \subseteq \binom{U}{q}$, compute

$$x_{3q} = |\{(A, B, C) \in \mathcal{F}^3 : |A \cap B| = |A \cap C| = |B \cap C| = \emptyset\}|.$$
Counting disjoint triples

Problem (slightly simplified)

Let \( q \leq |U|/3 \). Given \( F \subseteq \binom{U}{q} \), compute

\[
x_{3q} = |\{(A, B, C) \in F^3 : |A \cap B| = |A \cap C| = |B \cap C| = \emptyset\}|.
\]

For a triple \( (A, B, C) \in F^3 \) define

\[
type(A, B, C) = |A \oplus B \oplus C|,
\]

where \( \oplus \) is the symmetric difference (xor).
Counting disjoint triples

Problem (slightly simplified)

Let $q \leq |U|/3$. Given $\mathcal{F} \subseteq \binom{U}{q}$, compute

$$x_{3q} = |\{(A, B, C) \in \mathcal{F}^3 : |A \cap B| = |A \cap C| = |B \cap C| = \emptyset\}|.$$

For a triple $(A, B, C) \in \mathcal{F}^3$ define

$$\text{type}(A, B, C) = |A \oplus B \oplus C|,$$

where $\oplus$ is the symmetric difference (xor).

Note: $|A \oplus B \oplus C| \equiv |A| + |B| + |C| = 3q \equiv q \pmod{2}$. 

Auxiliary indeterminates ($\lfloor \frac{3q}{2} \rfloor$ indeterminates in total)

For $j \equiv q \pmod{2}$, $j \in \{0, \ldots, 3q\}$,

$$x_j = \left\{ (A, B, C) \in \mathcal{F}^3 : |A \oplus B \oplus C| = j \right\}.$$
Counting disjoint triples

Problem (slightly simplified)

Let \( q \leq |U|/3 \). Given \( \mathcal{F} \subseteq \binom{U}{q} \), compute

\[
x_{3q} = |\{(A, B, C) \in \mathcal{F}^3 : |A \cap B| = |A \cap C| = |B \cap C| = \emptyset\}|.
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Auxiliary indeterminates (\(\lfloor \frac{3q}{2} \rfloor\) indeterminates in total)

For \( j \equiv q \pmod{2}, j \in \{0, \ldots, 3q\} \),

\[
x_j = \{(A, B, C) \in \mathcal{F}^3 : |A \oplus B \oplus C| = j\}
\]
First source of linear equations: Intersection Parity Counting
Intersection parity

For $W \in 2^U$ and $p = 0, 1$ let

$$T_p(W) = |\{(A, B, C) \in \mathcal{F}^3 : |(A \oplus B \oplus C) \cap W| \equiv p \ (\text{mod} \ 2)\}|$$
Linear equations

For \( W \in (U_{\leq i}) \) and \( p = 0, 1 \) let

\[
T_p(W) = |\{(A, B, C) \in F^3 : |(A \oplus B \oplus C) \cap W| \equiv p \ (\text{mod} \ 2)\}|
\]

Proof: Let \((A, B, C)\) be a triple of type \(j\).
We show that \((A, B, C)\) is counted \((n - 2j)^i\) times in the RHS.
Define \(v^p_i = |\{(d_1, \ldots, d_i) \in U^i : |(A \oplus B \oplus C) \cap \bigoplus \{d_r\}_{r=1}^i| \equiv p \ (\text{mod} \ 2)\}|\)
Then \((A, B, C)\) is counted \(v^0_i - v^1_i\) times in RHS.

\[
\begin{align*}
  v^0_i &= (n - j)v^0_{i-1} + jv^1_{i-1} + (n - j)v^1_{i-1} \\
  v^1_i &= jv^0_{i-1} + (n - j)v^1_{i-1} \\
  v^0_i - v^1_i &= (n - 2j)(v^0_{i-1} - v^1_{i-1}) \\
  v^0_0 &= 1, \quad v^1_0 = 0.
\end{align*}
\]
Computing $b_i = \sum_{d_1, \ldots, d_i \in U} T_0(\oplus \{d_r\}_{r=1}^i) - T_1(\oplus \{d_r\}_{r=1}^i)$ in $O^*(n^i + n^q)$ time

$$T_p(W) = |\{(A, B, C) \in \mathbb{F}^3 : |(A \oplus B \oplus C) \cap W| \equiv p \mod 2\}|$$

**Note:** It suffices to compute $T_p(W)$ for every $W \in \binom{U}{\leq i}$.

$$|(A \oplus B \oplus C) \cap W| = |(A \cap W) \oplus (B \cap W) \oplus (C \cap W)|$$

$$\equiv |A \cap W| + |B \cap W| + |C \cap W| \mod 2$$

**Observation:** $|(A \oplus B \oplus C) \cap W| \equiv 0$ iff

- all $|A \cap W|, |B \cap W|, |C \cap W|$ even or
- exactly one of $|A \cap W|, |B \cap W|, |C \cap W|$ even.

Let $n_p(W) = |\{S \in \mathbb{F} : |S \cap W| \equiv p \mod 2\}|$, for $p = 0, 1$. Then,

$$T_0(W) = n_0(W)^3 + 3n_0(W)n_1(W)^2; \quad T_1(W) = |\mathcal{F}|^3 - T_0(W).$$
Computing \( b_i = \sum_{d_1, \ldots, d_i \in U} T_0(\oplus \{d_r\}_{r=1}^i) - T_1(\oplus \{d_r\}_{r=1}^i) \) in 
\( O^*(n^i + n^q) \) time

- It suffices to compute \( T_p(W) \) for every \( W \in \binom{U}{\leq i} \).
- We showed

\[
T_0(W) = n_0(W)^3 + 3n_0(W)n_1(W)^2; \quad T_1(W) = |\mathcal{F}|^3 - T_0(W),
\]

where Let \( n_p(W) = |\{ S \in \mathcal{F} : |S \cap W| \equiv p \pmod{2} \}| \) for \( p = 0, 1 \).

\[
n_p(W) = \sum_{j \equiv p} |\{ S \in \mathcal{F} : |S \cap W| = j \}|
\]

- We find \( n_p(W) = \sum_{j \equiv p} |\{ S \in \mathcal{F} : |S \cap W| = j \}| \) for every \( j \) and \( W \in \binom{U}{\leq i} \) in \( O^*(|\downarrow \mathcal{F}| + |\downarrow \binom{n}{\leq i}|) = O^*(n^q + n^i) \) time using Fast Intersection Transform.

- From the values of \( n_p(W) \) we can compute any value of \( n_p(W) \) in \( O^*(1) \) time, so \( b_i \) can be found in \( O(n^i) \) time.
Corollary

The coefficients / constant term of the equation:

\[ \sum_{0 \leq j \leq 3q, \ j \equiv q \ (\mod 2)} (n - 2j)^i x_j = \sum_{d_1, \ldots, d_i \in U} T_0(\bigoplus \{d_r\}_{r=1}^i) - T_1(\bigoplus \{d_r\}_{r=1}^i) \]

can be computed in \( O^*(n^i + n^q) \) time, for any \( i \geq 0 \).

Observation

For the \( k \)-path application, \( q = k/3 \);
If we use only the first source we need \( \left\lfloor \frac{3q}{2} \right\rfloor + 1 = \left\lfloor k/2 \right\rfloor + 1 \) equations, which results in total \( O^*(n^{k/2}) \) time.
Second source of linear equations: computing $x_j$ for small $j$
Computing $x_j$ for small $j$: summing over all possible $A \oplus B$

Consider a triple $(A, B, C)$ of type $j$. Let $\ell = |A \oplus B|$.

Since $|A \oplus B| = |A \oplus B \oplus C \oplus C|$, $q - j \leq \ell \leq q + j$

Note that $\ell = |A| + |B| - 2|A \cap B| = 2q - 2|A \cap B| \equiv 0 \mod 2$.

Since $|A \oplus B \oplus C| = |A \oplus B| + |C| - 2|(A \oplus B) \cap C|$, $|A \oplus B \cap C| = \frac{\ell + q - j}{2}$

$$x_j = \sum_{q-j \leq \ell \leq q+j} \sum_{\substack{D \in \binom{U}{\ell} \\ \ell \equiv 0 \mod 2}} |D| \cdot |\{C \in \mathcal{F} : |D \cap C| = \frac{\ell + q - j}{2}\}|,$$

where $\ominus^{-1}(D) = \{(A, B) \in \mathcal{F}^2 : A \oplus B = D\}$
Computing $x_j$ for small $j$: summing over all possible $A \oplus B$

$$x_j = \sum_{q-j \leq \ell \leq q+j} \sum_{\ell \equiv 0 \pmod{2}} \left| \bigoplus^{-1}(D) \right| \cdot \left| \{ C \in \mathcal{F} : |D \cap C| = \frac{\ell+q-j}{2} \} \right|,$$

where $\bigoplus^{-1}(D) = \{(A, B) \in \mathcal{F}^2 : A \oplus B = D\}$

- $\left| \{ C \in \mathcal{F} : |D \cap C| = \frac{\ell+q-j}{2} \} \right| = \nu_{\mathcal{F}}(D, \frac{\ell+q-j}{2})$ can be computed for all $D \in \binom{U}{\ell}$ in $O^*(|\downarrow \binom{U}{\ell}| + |\downarrow \mathcal{F}|) = O^*(n^{q+j})$ time using fast intersection transform.

- How fast can we compute $\bigoplus^{-1}$?
Let $M$ be a matrix with

- rows indexed by sets $S \in \binom{U}{\ell/2}$,
- columns indexed by sets $X \in \binom{U}{q-\ell/2}$,
- $M_{SX} = [S \cup X \in \mathcal{F}]$.

Let $B = MM^T$. $B$ is indexed by sets $S \in \binom{U}{\ell/2}$.

$$B_{RS} = \sum_{X \in \binom{U}{q-\ell/2}} [R \cup X \in \mathcal{F}] \cdot [S \cup X \in \mathcal{F}] = |\{X \in \binom{U}{q-\ell/2} : R \cup X, S \cup X \in \mathcal{F}\}|.$$

Then,

$$|\oplus^{-1}(D)| = \sum_{R \cup S = D} B_{RS}$$

- $B$ can be computed in $O(\max\{n^{(\omega-2)\ell/2+q}, n^{\omega\ell/2}\})$ time.
- Hence, within the same time we can find $|\oplus^{-1}(D)|$ for all $D \in \binom{U}{\ell}$.
Computing $x_j$ for small $j$: summing over all possible $A \oplus B$

\[ x_j = \sum_{q-j \leq \ell \leq q+j} \sum_{D \in \binom{U}{\ell} : \ell \equiv 0 \pmod{2}} |\ominus^{-1}(D)| \cdot |\{C \in \mathcal{F} : |D \cap C| = \frac{\ell+q-j}{2}\}| , \]

where $\ominus^{-1}(D) = \{(A, B) \in \mathcal{F}^2 : A \oplus B = D\}$

- $|\{C \in \mathcal{F} : |D \cap C| = \frac{\ell+q-j}{2}\}|$ can be computed for all $D \in \binom{U}{\ell}$ in $O^*(|\downarrow\binom{U}{\ell}| + |\downarrow\mathcal{F}|) = O^*(n^{q+j})$ time using fast intersection transform.
- $|\ominus^{-1}(D)|$ can be computed in

\[
O(\max\{n^{(\omega-2)\ell/2+q}, n^{\omega \ell/2}\}) = O(\max\{n^{(\omega-2)(q+j)/2+q}, n^{\omega(q+j)/2}\}) = O(\max\{n^{\omega(q+j)/2-j}, n^{\omega(q+j)/2}\}) = O(n^{\omega(q+j)/2})
\]

time for all $D \in \binom{U}{\ell}$.

- Overall, $x_j$ can be computed in $O(n^{\omega(q+j)/2})$ time.
Corollary

The constant term of the equation:

\[
x_j = \sum_{q-j \leq \ell \leq q+j} \sum_{D \in \binom{U}{\ell} \,(\text{mod 2})} |\bigoplus^{-1}(D)| \cdot |\{ C \in \mathcal{F} : |D \cap C| = \frac{\ell + q - j}{2}\}|
\]

can be computed in \(O(n^{\omega(q+j)/2})\) time, for any \(j = 0, \ldots, \lfloor \frac{3q}{2} \rfloor\), \(j \equiv q\).
Setting up the system if linear equations

- Pick \( r \) equations from the first source:

\[
\sum_{0 \leq j \leq 3q} (n-2j)^i x_j = \sum_{j=q}^{r-1} T_0(\bigoplus\{d_r\}_{r=1}^i) - T_1(\bigoplus\{d_r\}_{r=1}^i); \quad i = 0, \ldots, r-1
\]

\( j \equiv q \pmod{2} \)

in \( \sum_{i=0}^{r-1} O^*(n^i + n^q) = O^*(n^r + n^q) \) time;

- Pick \( \left\lfloor \frac{3q}{2} \right\rfloor + 1 - r \) equations from the second source:

\[
x_j = \sum_{q-j \leq \ell \leq q+j} \sum_{D \in \left(\bigcup_{\ell=0}^{j=q} \right) D \in (U\ell)} |\bigoplus^{-1}(D)| \cdot \mathcal{F}(D, \frac{l+q-j}{2}), \quad j \equiv q
\]

in \( O^*(n^{\omega(q+2(\frac{3q}{2}-r))/2}) = O^*(n^{\omega(2q-r)}) \) time;

- Both running times meet at \( r = \frac{2\omega q}{1+\omega} \approx 1.408q \)
The missing piece: linear independence

\[ \{ (n - 2j)^i \} \] for \( i, j = 0, ..., r - 1 \)

Vandermonde matrix

Identity matrix

First source

Second source
Conclusion

Corollary

One can count disjoint triples of a family of $q$-subsets of $n$-element universe in $O^*(n^{1.408q})$ time.

By essentially the same arguments we can get...

Corollary

One can compute $\Delta(f, g, h) = \sum_{A, B, C \in \binom{U}{q}} f(A)g(B)h(C)$ in $O^*(n^{1.408q})$ time.

$|A \cap B| = |A \cap C| = |B \cap C| = \emptyset$

Corollary

One can count the number of $k$-paths in an $n$-vertex graph in $O^*(n^{0.47k})$ time.
After some improvements...

**Theorem (Björklund, Kaski, K. 2013)**

- One can count disjoint triples of a family of $q$-subsets of $n$-element universe in $O^*(n^{1.364q})$ time.
- One can compute
  \[
  \Delta(f, g, h) = \sum_{A, B, C \in \binom{U}{q}} f(A)g(B)h(C)
  \]
  \[
  \text{where } |A \cap B| = |A \cap C| = |B \cap C| = \emptyset
  \]
  in $O^*(n^{1.364q})$ time.
- One can count the number of $k$-paths in an $n$-vertex graph in $O^*(n^{0.455k})$ time.
- One can count the number of occurrences of a fixed $k$-vertex pathwidth $p$ subgraph in an $n$-vertex graph in $O^*(n^{0.455k+2p})$ time.