ADFOCS 2013
Parameterized Algorithms using Matroids – Exercise II

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1. In the d-Hitting Set problem, we are given a family $\mathcal{F}$ of sets of size d over a universe U and a positive integer k, and the problem is to test whether there exists a subset $X \subseteq U$ such that $|X| \leq k$ and for every set $F \in \mathcal{F}$, $F \cap X \neq \emptyset$. Obtain a $k^{\mathcal{O}(d)}$ kernel for the problem using the method of representative sets.

2. Let $A_1, \ldots, A_m$ be p-element sets and $B_1, \ldots, B_m$ be q-element sets such that $A_i \cap B_j = \emptyset$ if and only of $i = j$.

   (a) Show that $m \leq 2^{p+q}$. (Hint: Think uniform random partition of $U = \cup_{i=1}^{m}(A_i \cup B_i)$.)

   (b) Show that $m \leq \binom{p+q}{p}$. (Hint: Think of permutations of U.)

   (c) Show that the bound of $\binom{p+q}{p}$ on m is tight.

   (d) Let $\mathcal{S} = \{S_1, \ldots, S_t\}$ be a family of p element sets. Using the above exercises show that the size of q-representative family is upper bounded by $\binom{p+q}{p}$.

3. Let $M = (U, \mathcal{I})$ be a matroid and let $\mathcal{S}$ be a p-uniform family of subsets of E. Show that if $\mathcal{S}' \subseteq_{rep}^{q} \mathcal{S}$ and $\widehat{\mathcal{S}} \subseteq_{rep}^{q} \mathcal{S}'$, then $\widehat{\mathcal{S}} \subseteq_{rep}^{q} \mathcal{S}$. (If $\widehat{\mathcal{S}} \subseteq \mathcal{S}$ is q-representative for $\mathcal{S}$ we write $\widehat{\mathcal{S}} \subseteq_{rep}^{q} \mathcal{S}$.)

4. Let $M = (U, \mathcal{I})$ be a matroid and let $\mathcal{S}$ be a p-uniform family of subsets of E. Show that if $\mathcal{S} = \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_\ell$ and $\widehat{\mathcal{S}}_i \subseteq_{rep}^{q} \mathcal{S}_i$, then $\cup_{i=1}^{\ell} \widehat{\mathcal{S}}_i \subseteq_{rep}^{q} \mathcal{S}$.

5. Let G be a connected graph on $2n$ vertices and $\mathcal{L}$ be a family of forests of G of size n (that is, the number of edges is n). Let $\widehat{\mathcal{L}} \subseteq \mathcal{L}$ be a family of forests such that for any forest F of size $n-1$, if there exists a forest $X \in \mathcal{L}$ such that $F \cup X$ is a spanning tree of G, then there exists a forest $\widehat{X} \in \widehat{\mathcal{L}}$ such that $F \cup \widehat{X}$ is a spanning tree of G. Could you give a non-trivial upper bound on the size of $|\widehat{\mathcal{L}}|$ (like some c^n)?