Parameterized Algorithms using Matroids
Lecture II: Representative Sets

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Problems we would be interested in...

Vertex Cover

**Input:** A graph $G = (V, E)$ and a positive integer $k$.

**Parameter:** $k$

**Question:** Does there exist a subset $V' \subseteq V$ of size at most $k$ such that for every edge $(u, v) \in E$ either $u \in V'$ or $v \in V'$?

Hamiltonian Path

**Input:** A graph $G = (V, E)$

**Question:** Does there exist a path $P$ in $G$ that spans all the vertices?

Path

**Input:** A graph $G = (V, E)$ and a positive integer $k$.

**Parameter:** $k$

**Question:** Does there exist a path $P$ in $G$ of length at least $k$?
REPRESENTATIVE SETS

Why, What and How.
REPRESENTATIVE SETS

Why, What and How.
Dynamic Programming for Hamiltonian Path
Example: \( V[\text{Paths of length } i \text{ ending at } v_j] \)

SETS, NOT SEQUENCES.

Two paths that use the same set of vertices but visit them in different orders are equivalent.

Valid:

Invalid:

Potentially storing \((n^i)\) sets.
Example:

\[ V[Paths of length i ending at v_j] \]

SETS, NOT SEQUENCES.

Two paths that use the same set of vertices but visit them in different orders are equivalent.

\[ V[Paths of length (i - 1) ending at u, avoiding v_j] \]

Valid:

Invalid:

Potentially storing \( n_i \) sets.
Example:

\[ V = \{ \text{Paths of length } i \text{ ending at } v_j \} \]

SETS, NOT SEQUENCES.

Two paths that use the same set of vertices but visit them in different orders are equivalent.

\[ V = \{ \text{Paths of length } (i - 1) \text{ ending at } u, \text{ avoiding } v_j \} \]

Valid:

Invalid:

Potentially storing \( (n_i) \) sets.
Example:

\[ V[Paths \text{ of length } i \text{ ending at } v_j] = V[Paths \text{ of length } (i-1) \text{ ending at } u, \text{ avoiding } v_j] \]

Valid:  
Invalid:  

Potentially storing \((n^i)\) sets.
**Ham-Path**

Example:

\[ V \left[ \text{Paths of length } i \text{ ending at } v_j \right] = V \left[ \text{Paths of length } (i - 1) \text{ ending at } u, \text{ avoiding } v_j \right] \]

Valid:

Invalid:

Potentially storing \((n^i)\) sets.
Example:

\[ V[\text{Paths of length } i \text{ ending at } v_j] \]

Two paths that use the same set of vertices but visit them in different orders are equivalent.

Valid:

Invalid:

Potentially storing \((n^n)\) sets.
Example:

\[ V[\text{Paths of length } i \text{ ending at } v_j] \]
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Sets, not sequences. Two paths that use the same set of vertices but visit them in different orders are equivalent.
Example:

\[ V[\text{Paths of length } i \text{ ending at } v_j] \]

Sets, not sequences.

POTENTIALLY STORING \((n^i)\) SETS.
Example:

\[ \text{Ham-Path} \]

SETS, NOT SEQUENCES.

\[ \forall \text{Paths of length } i \text{ ending at } v_j \]

\[ V \]

\[ v_1 \]

\[ \vdots \]

\[ v_j \]

\[ \vdots \]

\[ v_n \]
Example:

```
1 2 3 \ldots i \ldots n-1 n
```

\( \{v_1\} \)

\[ \vdots \]

\( \{v_j\} \)

\[ \vdots \]

\( \{v_n\} \)

\mathcal{V}[\text{Paths of length } i \text{ ending at } v_j] \\
\text{SETS, NOT SEQUENCES.}
Example:

\[
\begin{array}{cccccccc}
1 & 2 & 3 & \cdots & i & \cdots & n-1 & n \\
\end{array}
\]

\[v_1\]

\[\vdots\]

\[v_j\]

\[\vdots\]

\[v_n\]

SETS, NOT SEQUENCES.

\[V[\text{Paths of length } i \text{ ending at } v_j]\]
Example: \(V[\text{Paths of length } i \text{ ending at } v_j]\)

Sets, not sequences.

Two paths that use the same set of vertices but visit them in different orders are equivalent.
**Ham-Path**

Example:

\[
V[\text{Paths of length } i \text{ ending at } v_j] = V[\text{Paths of length } (i - 1) \text{ ending at } u, \text{ avoiding } v_j].
\]

\[
\begin{array}{ccccccc}
1 & 2 & 3 & \cdots & i & \cdots & n-1 & n \\
\end{array}
\]

\[
v_1
\]

\[
\vdots
\]

\[
\bigcirc v_j
\]

\[
v_n
\]
**Ham-Path**

Example:

\[ V[\text{Paths of length } i \text{ ending at } v_j] \]

\[ = V[\text{Paths of length } (i - 1) \text{ ending at } u, \text{ avoiding } v_j.] \]

\[ u \in N(v_j) \]

\[ v_1 \]

\[ \vdots \]

\[ v_j \]

\[ \vdots \]

\[ v_n \]
Valid:

\[
\begin{array}{cccccccc}
1 & 2 & 3 & \cdots & i & \cdots & n-1 & n \\
\end{array}
\]

\[v_1 \]
\[\vdots\]
\[v_j \]
\[V[\text{Paths of length } i \text{ ending at } v_j]\]
\[= V[\text{Paths of length } (i-1) \text{ ending at } u, \text{ avoiding } v_j.]\]
\[u \in N(v_j)\]
Invalid:

\[ V[\text{Paths of length } i \text{ ending at } v_j] = V[\text{Paths of length } (i - 1) \text{ ending at } u, \text{ avoiding } v_j] \quad u \in N(v_j) \]
Example:
\[ V[\text{Paths of length } i \text{ ending at } v_j] \]

Potentially storing \( \binom{n}{i} \) sets.

\[ v_1 \]
\[ \vdots \]
\[ v_j \]
\[ \vdots \]
\[ v_n \]

\[ = V[\text{Paths of length } (i - 1) \text{ ending at } u, \text{ avoiding } v_j.] \]
\[ u \in N(v_j) \]
Let us now turn to $k$-Path.

To find paths of length at least $k$, we may simply use the DP table for Hamiltonian Path restricted to the first $k$ columns.
K-Path

Worst case running time: $O^* \left( \binom{n}{k} \right)$
Worst case running time: $O^*(n^k)$
Do we really need to store all these sets?
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In the $i^{th}$ column, we are storing paths of length $i$. 
Do we really need to store all these sets?

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Let $P$ be a path of length $k$. 
Do we really need to store all these sets?

In the $i^{th}$ column, we are storing paths of length $i$.

Let $P$ be a path of length $k$.

There may be several paths of length $i$ that “latch on” to the last $(k - i)$ vertices of $P$. 
Do we really need to store all these sets?

In the $i^{th}$ column, we are storing paths of length $i$.

Let $P$ be a path of length $k$.

There may be several paths of length $i$ that “latch on” to the last $(k - i)$ vertices of $P$.

We need to store just one of them.
Example.
Example.

Suppose we have a path $P$ on seven edges.
Example.

Suppose we have a path $P$ on seven edges.

Consider it broken up into the first four and the last three edges.
A Fixed Future ($\nu_{i+1} - \nu_k$).
A Fixed Future (v + 1 - v k).

The Possibilities for Partial Solutions Compatible with v i + 1 - v k.
A Fixed Future ($v_{i+1}$). The Possibilities for Partial Solutions Compatible with $v_{i+1} - v_k$. 
A Fixed Future ($v_{i+1} - v_k$).

The Possibilities for Partial Solutions Compatible with $v_{i+1} - v_k$. 
A Fixed Future ($v_{i+1} - v_k$).

The Possibilities for Partial Solutions Compatible with $v_{i+1} - v_k$. 
A Fixed Future \((v_{i+1} - \cdots - v_k)\).
The Possibilities for Partial Solutions Compatible with $v_{i+1} \cdots v_k$.

A Fixed Future ($v_{i+1} \cdots v_k$).
Let's try a different example.
The Possibilities for Partial Solutions Compatible with $\nu_{i+1} - \cdots - \nu_k$.

A Fixed Future ($\nu_{i+1} - \cdots - \nu_k$).
Here's one more example:
The Possibilities for Partial Solutions Compatible with $v_{i+1} - \cdots - v_k$.

A Fixed Future ($v_{i+1} - \cdots - v_k$).
For any possible ending of length \((k - i)\), we want to be sure that we store at least one among the possibly many “prefixes”.
For any possible ending of length $(k - i)$, we want to be sure that we store at least one among the possibly many “prefixes”.

This could also be $\binom{n}{k-i}$.
For any possible ending of length \((k - i)\), we want to be sure that we store at least one among the possibly many “prefixes”.

This could also be \(\binom{n}{k-i}\).

The hope for “saving” comes from the fact that a single path of length \(i\) is potentially capable of being a prefix to several distinct endings.
For example...
REPRESENTATIVE SETS

Why, What and How.
Partial solutions: paths of length $j$ ending at $v_i$
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A “small” representative family.
Partial solutions: paths of length $j$ ending at $v_i$

A “small” representative family.
Partial solutions: paths of length $j$ ending at $v_i$

A “small” representative family.
Partial solutions: paths of length $j$ ending at $v_i$

If:

Then:

A “small” representative family.
Partial solutions: paths of length $j$ ending at $v_i$.

If: $v_i$ is reached from $(k - j)$ vertices.

Then: We would like to store at least one path of length $j$ that serves the same purpose.
Partial solutions: paths of length $j$ ending at $v_i$
If:

\[ v_i \]

Then:

A "small" representative family.

Partial solutions: paths of length \( j \) ending at \( v_i \)
Partial solutions: paths of length $j$ ending at $v_i$.

A “small” representative family.

We would like to store at least one path of length $j$ that serves the same purpose.
Given: A (BIG) family \( \mathcal{F} \) of \( p \)-sized subsets of \([n]\).

\[ S_1, S_2, \ldots, S_t \]
Given: A (BIG) family $\mathcal{F}$ of $p$-sized subsets of $[n]$.

$S_1, S_2, \ldots, S_t$

Want: A (small) subfamily $\hat{\mathcal{F}}$ of $\mathcal{F}$ such that:
Given: A (BIG) family $\mathcal{F}$ of $p$-sized subsets of $[n]$.

$S_1, S_2, \ldots, S_t$

Want: A (small) subfamily $\hat{\mathcal{F}}$ of $\mathcal{F}$ such that:

For any $X \subseteq [n]$ of size $(k - p)$,

if there is a set $S$ in $\mathcal{F}$ such that $X \cap S = \emptyset$,
then there is a set $\hat{S}$ in $\hat{\mathcal{F}}$ such that $X \cap \hat{S} = \emptyset$. 
Given: A (BIG) family $\mathcal{F}$ of $p$-sized subsets of $[n]$.

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then there is a set $\hat{S}$ in $\hat{\mathcal{F}}$ such that $X \cap \hat{S} = \emptyset$.

The “second half” of a solution — can be any subset.
Given: A (BIG) family $\mathcal{F}$ of $p$-sized subsets of $[n]$.

$S_1, S_2, \ldots, S_t$

Want: A (small) subfamily $\hat{\mathcal{F}}$ of $\mathcal{F}$ such that:

For any $X \subseteq [n]$ of size $(k - p)$,

if there is a set $S$ in $\mathcal{F}$ such that $X \cap S = \emptyset$,
then there is a set $\hat{S}$ in $\hat{\mathcal{F}}$ such that $X \cap \hat{S} = \emptyset$.

This is a valid patch into $X$. 
Given: A (BIG) family $\mathcal{F}$ of $p$-sized subsets of $[n]$.

$S_1, S_2, \ldots, S_t$

Want: A (small) subfamily $\hat{\mathcal{F}}$ of $\mathcal{F}$ such that:

For any $X \subseteq [n]$ of size $(k - p)$,

if there is a set $S$ in $\mathcal{F}$ such that $X \cap S = \emptyset$,
then there is a set $\hat{S}$ in $\hat{\mathcal{F}}$ such that $X \cap \hat{S} = \emptyset$.

This is a guaranteed replacement for $S$. 
Given: A (BIG) family $\mathcal{F}$ of $p$-sized subsets of $[n]$.

$S_1, S_2, \ldots, S_t$

Want: A (small) subfamily $\mathcal{\hat{F}}$ of $\mathcal{F}$ such that:

For any $X \subseteq [n]$ of size $(k - p)$,

if there is a set $S$ in $\mathcal{F}$ such that $X \cap S = \emptyset$,
then there is a set $\hat{S}$ in $\mathcal{\hat{F}}$ such that $X \cap \hat{S} = \emptyset$. 
Given: A $\leq \binom{n}{p}$ family $F$ of $p$-sized subsets of $[n]$.

$S_1, S_2, \ldots, S_t$

Want: A (small) subfamily $\hat{F}$ of $F$ such that:

For any $X \subseteq [n]$ of size $(k - p)$, if there is a set $S$ in $F$ such that $X \cap S = \emptyset$, then there is a set $\hat{S}$ in $\hat{F}$ such that $X \cap \hat{S} = \emptyset$. 
Given: A \subseteq \binom{n}{p} family \mathcal{F} of p-sized subsets of [n].

\[ S_1, S_2, \ldots, S_t \]

Known: \exists \binom{k}{p} subfamily \mathcal{F} of \mathcal{F} such that:

For any \( X \subseteq [n] \) of size \((k - p)\),

if there is a set \( S \) in \( \mathcal{F} \) such that \( X \cap S = \emptyset \),
then there is a set \( \hat{S} \) in \( \mathcal{F} \) such that \( X \cap \hat{S} = \emptyset \).

Bolobás, 1965.
Given: A matroid $(M, I)$, and a family of $p$-sized subsets from $I$:

$$S_1, S_2, \ldots, S_t$$
Given: A matroid \((M, I)\), and a family of \(p\)-sized subsets from \(I\):

\[ S_1, S_2, \ldots, S_t \]

Want: A subfamily \(\tilde{\mathcal{F}}\) of \(\mathcal{F}\) such that:
Given: A matroid \((M, I)\), and a family of \(p\)-sized subsets from \(I\):

\[ S_1, S_2, \ldots, S_t \]

Want: A subfamily \(\hat{\mathcal{F}}\) of \(\mathcal{F}\) such that:

For any \(X \subseteq [n]\) of size at most \(q\),

if there is a set \(S\) in \(\mathcal{F}\) such that \(X \cap S = \emptyset\) and \(X \cup S \in I\),
then there is a set \(\hat{S}\) in \(\hat{\mathcal{F}}\) such that \(X \cap \hat{S} = \emptyset\) and \(X \cup \hat{S} \in I\).
Given: A matroid \((M, I)\), and a family of \(p\)-sized subsets from \(I\):

\[ S_1, S_2, \ldots, S_t \]

There is a subfamily \(\hat{F}\) of \(F\) of size at most \(\binom{p+q}{p}\) such that:

For any \(X \subseteq [n]\) of size at most \(q\),

if there is a set \(S\) in \(F\) such that \(X \cap S = \emptyset\) and \(X \cup S \in I\),

then there is a set \(\hat{S}\) in \(\hat{F}\) such that \(X \cap \hat{S} = \emptyset\) and \(X \cup \hat{S} \in I\).

Lovász, 1977
Given: A matroid $(M, I)$, and a family of $p$-sized subsets from $I$:

$$S_1, S_2, \ldots, S_t$$

There is an efficiently computable subfamily $\hat{F}$ of $F$ of size at most ${\binom{p+q}{p}}$ such that:

For any $X \subseteq [n]$ of size at most $q$,

if there is a set $S$ in $F$ such that $X \cap S = \emptyset$ and $X \cup S \in I$,
then there is a set $\hat{S}$ in $\hat{F}$ such that $X \cap \hat{S} = \emptyset$ and $X \cup \hat{S} \in I$.

Márx (2009) and Fomin, Lokshtanov, Saurabh (2013)
Summary.

We have at hand a $p$-uniform collection of independent sets, $\mathcal{F}$ and a number $q$. Let $X$ be any set of size at most $q$. For any set $S \in \mathcal{F}$, if:

a. $X$ is disjoint from $S$, and

b. $X$ and $S$ together form an independent set,

then a $q$-representative family $\hat{\mathcal{F}}$ contains a set $\hat{S}$ that is:

a. disjoint from $X$, and

b. forms an independent set together with $X$.

Such a subfamily is called a $q$-representative family for the given family.
REPRESENTATIVE SETS

Back to Why.
Worst case running time: $\Theta^*\left(\binom{n}{k}\right)$
Worst case running time: $O^{\star}(n^k)$.
Worst case running time: \( O^{\ast}(n^k) \)

[RECALL] \( \binom{n}{k} \)

Representative Set Computation
Worst case running time: $O^\star((n^k))$.

[RECALL] $(\binom{n}{k})$

Representative Set Computation

$(\binom{k}{i})$
Worst case running time: $O^{\star}(n^k)$

Not so fast! $(n^k)$

Representative Set Computation

$v_1$

$v_i$

$v_j$

$v_n$
Worst case running time: $O(\star ((n \choose k)))$

Not so fast! $(n \choose k)$ is too big!

Representative Set Computation

$(k \choose i)$
We are going to compute representative families at every intermediate stage of the computation.
We are going to compute representative families at every intermediate stage of the computation.

For instance, in the $i^{th}$ column, we are storing $i$-uniform families. Before moving on to column $(i + 1)$, we compute $(k - i)$-representative families.

This keeps the sizes small as we go along.
Representative Set Computation

\[ v_1, \ldots, v_j, \ldots, v_{k-1}, v_k \]

Worst case running time:

\[ \mathcal{O}(n^k) \]
Representative Set Computation

\[ v_1, \ldots, v_{i-1}, v_i, \ldots, v_{k-1}, v_k \]

Worst case running time:

\[ O^{\ast}(n^k) \]
Worst case running time: $O(^{*}((n^k)))$
Worst case running time: $O^\star((n^k))$.
### Representative Set Computation

<table>
<thead>
<tr>
<th>$v_1$</th>
<th>$(k)_1$</th>
<th>$v_j$</th>
<th>$(k)_2$</th>
<th>$\cdots$</th>
<th>$(k)_{i-1}$</th>
<th>$v_n$</th>
<th>$(k)_n$</th>
</tr>
</thead>
</table>

- **Worst case running time:** $O^{\star}((n^k))$
Worst case running time: \( O(\star ((n^k))) \).

**Representative Set Computation**

\[ n_1 (k_1) \quad n_2 (k_2) \quad n_3 (k_3) \quad \cdots \quad n_i (k_i) \quad \cdots \quad n_{k-1} (k_{k-1}) \quad n_k (k_k) \]
Worst case running time: $O(\ast(n^k))$.

Representative Set Computation

$\nu_1$  $(^k_1)$  $(^k_2)$  $(^k_3)$  ...  $(^k_{i-1})n$

$\nu_j$  $(^k_1)$  $(^k_2)$  $(^k_3)$  ...  $(^k_{i-1})n$

$\nu_n$  $(^k_1)$  $(^k_2)$  $(^k_3)$  ...  $(^k_{i-1})n$
Representative Set Computation

$\nu_1 \quad (k_1) \quad (k_2) \quad (k_3) \quad \ldots \quad (k_i) \quad \ldots \quad (k_{i-1}) \quad (k)$

Worst case running time: $O(\star((n^k)))$
Representative Set Computation

$\nu_1 \quad \binom{k}{1} \quad \binom{k}{2} \quad \binom{k}{3} \quad \ldots \quad \binom{k}{i} \quad \ldots \quad 2^k n$

Worst case running time: $O(^{\ast}(n^k))$
Worst case running time: $O(\star (n^k))$.
Let $\mathcal{P}_i^j$ be the set of all paths of length $i$ ending at $v_j$.

It can be shown that the families thus computed at the $i^{th}$ column, $j^{th}$ row are indeed $(k - i)$-representative families for $\mathcal{P}_i^j$.

The correctness is implicit in the notion of a representative family.
REPRESENTATIVE SETS

A Different Why.
Vertex Cover
Can you delete $k$ vertices to kill all edges?
Vertex Cover
Can you delete $k$ vertices to kill all edges?
Let \((G = (V, E), k)\) be an instance of Vertex Cover.

Note that \(E\) can be thought of as a 2-uniform family over the ground set \(V\).
Let \((G = (V, E), k)\) be an instance of Vertex Cover.

Note that \(E\) can be thought of as a 2-uniform family over the ground set \(V\).

---

**Goal: Kernelization.**

In this context, we are asking if there is a small subset \(X\) of the edges such that

\[ G[X] \text{ is a YES-instance } \iff G \text{ is a YES-instance.} \]
Note: If $G$ is a YES-instance, then $G[X]$ is a YES-instance for any subset $X \subseteq E$. 
Note: If $G$ is a YES-instance, then $G[X]$ is a YES-instance for any subset $X \subseteq E$.

We get one direction for free!
Note: If $G$ is a YES-instance, then $G[X]$ is a YES-instance for any subset $X \subseteq E$.

We get one direction for free!

It is the **NO-instances** that we have to worry about preserving.
Note: If $G$ is a YES-instance, then $G[X]$ is a YES-instance for any subset $X \subseteq E$.

We get one direction for free!

It is the **NO-instances** that we have to worry about preserving.

What is a NO-instance?
If $G$ is a NO-instance:

For any subset $S$ of size at most $k$, there is an edge that is disjoint from $S$. 

Ring a bell?
If $G$ is a NO-instance:

For any subset $S$ of size at most $k$, there is an edge that is disjoint from $S$.

Ring a bell?
Recall.

We have at hand a $p$-uniform collection of independent sets, $\mathcal{F}$ and a number $q$. Let $X$ be any set of size at most $q$. For any set $S \in \mathcal{F}$, if:

a. $X$ is disjoint from $S$, and

b. $X$ and $S$ together form an independent set,

then a $q$-representative family contains a set $\hat{S}$ that is:

a. disjoint from $X$, and

b. forms an independent set together with $X$.

Such a subfamily is called a $q$-representative family for the given family.
Claim: A $k$-representative family for $E$ is in fact an $O(k^2)$ kernel for vertex cover.
Is there a Vertex Cover of size at most $k$?
\[ E(G) = \{ e_1, e_2, \ldots, e_m \} \]

Is there a Vertex Cover of size at most k?
$E(G) = \{e_1, e_2, \ldots, e_m\}$

$k$-Representative Family

$\{f_1, f_2, \ldots, f_r\}$

Is there a Vertex Cover of size at most $k$?
Is there a Vertex Cover of size at most $k$?
E(G) = \{e_1, e_2, \ldots, e_m\}

\{f_1, f_2, \ldots, f_r\}

\Theta(k^2)

Is there a Vertex Cover of size at most k?
Let us show that if $G[X]$ is a YES-instance, then so is $G$. 
Let us show that if \( G[X] \) is a YES-instance, then so is \( G \).

This time, by contradiction.
Try the solution for $\mathcal{G}[\mathcal{X}]$ on $\mathcal{G}$.

Suppose there is an uncovered edge. Since $\mathcal{X}$ is a $k$-representative family, for ANY $S \subseteq V$, where $|S| \geq k$

if there is a set $e$ in $E$ such that $e \cap S = H$

then there is a set $p \in \mathcal{X}$ such that $p \cap S = H$.

Note that the green edges denote $\mathcal{G}[\mathcal{X}]$.

Contradiction!
Try the solution for $G[X]$ on $G$. 

Contradiction!
Suppose there is an uncovered edge.
Since $X$ is a $k$-representative family, for ANY $S \subseteq V$, where $|S| \leq k$: 
Since $X$ is a $k$-representative family, for ANY $S \subseteq V$, where $|S| \leq k$:

if there is a set $e$ in $E$ such that $e \cap S = \emptyset$, 

Contradiction!
Since $X$ is a $k$-representative family, for ANY $S \subseteq V$, where $|S| \leq k$: 

if there is a set $e$ in $E$ such that $e \cap S = \emptyset$, then there is a set $\hat{e}$ in $X$ such that $\hat{e} \cap S = \emptyset$. 
Since $X$ is a $k$-representative family, for ANY $S \subseteq V$, where $|S| \leq k$:

if there is a set $e$ in $E$ such that $e \cap S = \emptyset$,
then there is a set $\widehat{e}$ in $X$ such that $\widehat{e} \cap S = \emptyset$.

Note that the green edges denote $G[X]$. 
Since $X$ is a $k$-representative family, for ANY $S \subseteq V$, where $|S| \leq k$:

if there is a set $e$ in $E$ such that $e \cap S = \emptyset$,
then there is a set $\hat{e}$ in $X$ such that $\hat{e} \cap S = \emptyset$.

Note that the green edges denote $G[X]$. 
Since $X$ is a $k$-representative family, for ANY $S \subseteq V$, where $|S| \leq k$:

if there is a set $e$ in $E$ such that $e \cap S = \emptyset$, then there is a set $\tilde{e}$ in $X$ such that $\tilde{e} \cap S = \emptyset$.

Note that the green edges denote $G[X]$. 

Contradiction!
Since $X$ is a $k$-representative family, for ANY $S \subseteq V$, where $|S| \leq k$:

if there is a set $e$ in $E$ such that $e \cap S = \emptyset$,
then there is a set $\hat{e}$ in $X$ such that $\hat{e} \cap S = \emptyset$.

Note that the green edges denote $G[X]$. 
Since $X$ is a $k$-representative family, for ANY $S \subseteq V$, where $|S| \leq k$:

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if there is a set $e$ in $E$ such that $e \cap S = \emptyset$, then there is a set $\hat{e}$ in $X$ such that $\hat{e} \cap S = \emptyset$.

Note that the green edges denote $G[X]$. 
Since \( X \) is a \( k \)-representative family, for ANY \( S \subseteq V \), where \( |S| \leq k \):

- if there is a set \( e \) in \( E \) such that \( e \cap S = \emptyset \),
- then there is a set \( \hat{e} \) in \( X \) such that \( \hat{e} \cap S = \emptyset \).

Note that the green edges denote \( G[X] \).

Contradiction!
A $k$-representative family for $E(G)$ is in fact an $O(k^2)$ instance kernel for Vertex Cover!
REPRESENTATIVE SETS

Why, What and How.
Notation

\[ \text{Det}(M) : \begin{bmatrix} M \end{bmatrix} \]

Let \( M \) be a \( m \times n \) matrix, and let \( I \subseteq [m], J \subseteq [n] \).

\( M[I, J] : M \) restricted to rows indexed by \( I \) and columns indexed by \( J \)

\( M[\star, J] : M \) restricted to all rows and columns indexed by \( J \)

\( M[I, \star] : M \) restricted to rows indexed by \( I \) and all columns
STANDARD LAPLACE EXPANSION
\[
\begin{bmatrix}
\ \ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\
\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\
\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\
\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\
\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\
\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66}
\end{bmatrix}
\]

Fix a row and expand along the columns.
\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\
  a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\
  a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\
  a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\
  a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\
  a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66}
\end{bmatrix}
\]

Fix a row and expand along the columns.
Fix a row and expand along the columns.
Fix a row and expand along the columns.

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\
  a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\
  a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\
  a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\
  a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\
  a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66}
\end{bmatrix}
\]

\[
\begin{bmatrix}
  a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\
  a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\
  a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\
  a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\
  a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\
  a_{62} & a_{63} & a_{64} & a_{65} & a_{66}
\end{bmatrix}
\]

\[
\begin{bmatrix}
  a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\
  a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\
  a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\
  a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\
  a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\
  a_{62} & a_{63} & a_{64} & a_{65} & a_{66}
\end{bmatrix}
\]

\[
\begin{bmatrix}
  a_{11} & a_{13} & a_{14} & a_{15} & a_{16} \\
  a_{21} & a_{23} & a_{24} & a_{25} & a_{26} \\
  a_{31} & a_{33} & a_{34} & a_{35} & a_{36} \\
  a_{41} & a_{43} & a_{44} & a_{45} & a_{46} \\
  a_{51} & a_{53} & a_{54} & a_{55} & a_{56} \\
  a_{61} & a_{63} & a_{64} & a_{65} & a_{66}
\end{bmatrix}
\]
Fix a row and expand along the columns.
Fix a row and expand along the columns.
Fix a row and expand along the columns.
Fix a row and expand along the columns.
GENERALIZED LAPLACE EXPANSION
\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\
a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\
a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\
a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66}
\end{bmatrix}
\]

\[
\text{Det}(A) = \sum_{I \subseteq [n], |I| = |J|} \text{Det}(A[I, J]) \cdot \text{Det}(A[\bar{I}, \bar{J}]) \cdot (-1)^{\sum I + \sum J}
\]
Fix a set of columns, $J \subseteq [6]$.

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\
  a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\
  a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\
  a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\
  a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\
  a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66}
\end{bmatrix}
\]

\[
\text{Det}(A) = \sum_{I \subseteq [n], |I|=|J|} \text{Det}(A[I, J]) \cdot \text{Det}(A[\bar{I}, \bar{J}]) \cdot (-1)^{|I+J|}
\]
Fix a set of columns, \( J \subseteq [6] \).

Iterate over all \( I \subseteq [6] \) such that \(|I| = |J|\).

\[
\text{Det}(A) = \sum_{I \subseteq [n], |I| = |J|} \text{Det}(A[I, \bar{J}]) \cdot \text{Det}(A[I, J]) \cdot (-1)^{\sum I + \sum J}
\]
Fix a set of columns, $J \subseteq [6]$.

Iterate over all $I \subseteq [6]$ such that $|I| = |J|$.

$$\det(A_{\bar{I}, \bar{J}}).$$

$$\det(A) = \sum_{I \subseteq [n], |I| = |J|} \det(A[I, J]) \cdot \det(A[I, J]) \cdot (-1)^{\sum I + \sum J}$$
Fix a set of columns, $J \subseteq [6]$. 

Iterate over all $I \subseteq [6]$ such that $|I| = |J|$.

$$\text{Det}(A[I, J]) = \sum_{I \subseteq [n], |I|=|J|} \text{Det}(A[I, J]) \cdot \text{Det}(A[I, J]) \cdot (-1)^{\sum I + \sum J}$$
Fix a set of columns, $J \subseteq [6]$.

Iterate over all $I \subseteq [6]$ such that $|I| = |J|$.

$$
\text{Det}(A_{[I, J]}) \cdot \text{Det}(A_{[I, J]}) \cdot (-1)^{\sum I + \sum J}$$
Fix a set of columns, \( J \subseteq [6] \).

Iterate over all \( I \subseteq [6] \) such that \( |I| = |J| \).

\[
\text{Det}(A[I, J]).
\]

\[
\text{Det}(A[I', J]).
\]

\[
\text{Det}(A[I', J]).
\]

\[
\text{Det}(A) = \sum_{I \subseteq [n], |I| = |J|} \text{Det}(A[I, J]) \cdot \text{Det}(A[I', J]) \cdot (-1)^{\sum I + \sum J}
\]
Fix a set of columns, \( J \subseteq [6] \).

Iterate over all \( I \subseteq [6] \) such that \(|I| = |J|\).

\[
\text{Det}(A[I, J]).
\]

\[
(-1)^{(1+3+6)+(1+2+3)}
\]

\[
\text{Det}(A[I, J]).
\]

\[
\text{Det}(A) = \sum_{I \subseteq [n], |I| = |J|} \text{Det}(A[I, \bar{J}]) \cdot \text{Det}(A[I, J]) \cdot (-1)^{\sum I + \sum J}
\]
Fix a set of columns, $J \subseteq [6]$. 

Iterate over all $I \subseteq [6]$ such that $|I| = |J|$. 

$$\begin{vmatrix}
 a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\
 a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\
 a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\
 a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\
 a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\
 a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66}
\end{vmatrix}$$

$$\begin{vmatrix}
 a_{11} & a_{13} & a_{16} \\
 a_{21} & a_{23} & a_{26} \\
 a_{41} & a_{43} & a_{46} \\
 a_{51} & a_{54} & a_{55} \\
 a_{62} & a_{64} & a_{65}
\end{vmatrix}$$

$$(-1)^{(1+3+6)+(1+2+4)}$$

$\text{Det}(A[I, J])$. 

$\text{Det}(A[I', J])$. 

$$\text{Det}(A) = \sum_{I \subseteq [n], |I| = |J|} \text{Det}(A[I, J]) \cdot \text{Det}(A[I', J]) \cdot (-1)^{\sum I + \sum J}$$
Fix a set of columns, $J \subseteq [6]$. 

Iterate over all $I \subseteq [6]$ such that $|I| = |J|$. 

\[
\text{Det}(\bar{A}) = \sum_{I \subseteq [n], |I| = |J|} \text{Det}(A[I, \bar{J}]) \cdot \text{Det}(A[I, J]) \cdot (-1)^{\sum I + \sum J}
\]
Fix a set of columns, $J \subseteq [6]$.

Iterate over all $I \subseteq [6]$ such that $|I| = |J|$.

$$
\begin{vmatrix}
 a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\
 a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\
 a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\
 a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\
 a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\
 a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66}
\end{vmatrix}
$$

$$
\begin{vmatrix}
 a_{11} & a_{13} & a_{16} \\
 a_{21} & a_{23} & a_{26} \\
 a_{32} & a_{34} & a_{35} \\
 a_{42} & a_{44} & a_{45} \\
 a_{52} & a_{54} & a_{55} \\
 a_{61} & a_{63} & a_{66}
\end{vmatrix}
$$

$$
\begin{vmatrix}
 a_{11} & a_{13} & a_{16} \\
 a_{21} & a_{23} & a_{26} \\
 a_{32} & a_{34} & a_{35} \\
 a_{42} & a_{44} & a_{45} \\
 a_{52} & a_{54} & a_{55} \\
 a_{61} & a_{63} & a_{66}
\end{vmatrix}
$$

\[ \text{Det}(A[I, J]). \quad \text{Det}(A[I, J]). \]

\[ \text{Det}(A) = \sum_{I \subseteq [n], |I| = |J|} \text{Det}(A[I, \bar{J}]) \cdot \text{Det}(A[I, J]) \cdot (-1)^{I + \sum J} \]
Fix a set of columns, $J \subseteq [6]$.

Iterate over all $I \subseteq [6]$ such that $|I| = |J|$. 

$$\det(A) = \sum_{I \subseteq [n], |I|=|J|} \det(A[I, \bar{J}]) \cdot \det(A[I, J]) \cdot (-1)^{\sum I + \sum J}$$
Fix a set of columns, $J \subseteq [6]$.

Iterate over all $I \subseteq [6]$ such that $|I| = |J|$.

$$\det(A) = \sum_{I \subseteq [n], |I| = |J|} \det(A[I, J]) \cdot \det(A[I, J]) \cdot (-1)^{\sum I + \sum J}$$
Fix a set of columns, $J \subseteq [6]$.

Iterate over all $I \subseteq [6]$ such that $|I| = |J|$.

$$\det(A[I, J]) \cdot \det(A[I, J]) \cdot (-1)^{\sum I + \sum J}$$
Fix a set of columns, \( J \subseteq [6] \).

Iterate over all \( I \subseteq [6] \) such that \(|I| = |J|\).

\[
\begin{vmatrix}
| a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\
| a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\
| a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\
| a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\
| a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\
| a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
| a_{11} & a_{13} & a_{16} \\
| a_{22} & a_{24} & a_{25} \\
| a_{32} & a_{34} & a_{35} \\
| a_{62} & a_{64} & a_{65} \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
| a_{11} & a_{13} & a_{16} \\
| a_{41} & a_{43} & a_{46} \\
| a_{51} & a_{53} & a_{56} \\
\end{vmatrix}
\]

\[
(-1)^{(1+3+6)+(1+4+5)}
\]

\[
\text{Det}(A) = \sum_{I \subseteq [n], |I| = |J|} \text{Det}(A[I, \bar{J}]) \cdot \text{Det}(A[I, J]) \cdot (-1)^{\sum I + \sum J}
\]
Fix a set of columns, \( J \subseteq [6] \).

Iterate over all \( I \subseteq [6] \) such that \( |I| = |J| \).

\[
\text{Det}(A) = \sum_{I \subseteq [n], |I| = |J|} \text{Det}(A[I, \bar{J}]) \cdot \text{Det}(A[I, J]) \cdot (-1)^{\sum I + \sum J}
\]
Fix a set of columns, \( J \subseteq [6] \).

Iterate over all \( I \subseteq [6] \) such that \(|I| = |J|\).

\[
\begin{vmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} \\
\alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} & \alpha_{25} & \alpha_{26} \\
\alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} & \alpha_{35} & \alpha_{36} \\
\alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} & \alpha_{45} & \alpha_{46} \\
\alpha_{51} & \alpha_{52} & \alpha_{53} & \alpha_{54} & \alpha_{55} & \alpha_{56} \\
\alpha_{61} & \alpha_{62} & \alpha_{63} & \alpha_{64} & \alpha_{65} & \alpha_{66}
\end{vmatrix}
\]

\[
\begin{vmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{22} & \alpha_{24} & \alpha_{25} & \vdots & \vdots & \vdots \\
\alpha_{32} & \alpha_{34} & \alpha_{35} & \vdots & \vdots & \vdots \\
\alpha_{42} & \alpha_{44} & \alpha_{45} & \vdots & \vdots & \vdots \\
\alpha_{51} & \alpha_{53} & \alpha_{54} & \alpha_{55} & \vdots & \vdots \\
\alpha_{61} & \alpha_{63} & \alpha_{64} & \alpha_{65} & \alpha_{66} & \vdots \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
\alpha_{11} & \alpha_{13} & \alpha_{16} \\
\alpha_{12} & \alpha_{14} & \vdots \\
\alpha_{15} & \alpha_{17} & \alpha_{20} \\
\vdots & \vdots & \vdots \\
\alpha_{51} & \alpha_{53} & \alpha_{56} \\
\alpha_{52} & \alpha_{54} & \vdots \\
\alpha_{55} & \alpha_{58} & \alpha_{61} \\
\vdots & \vdots & \vdots \\
\alpha_{61} & \alpha_{63} & \alpha_{66} \\
\end{vmatrix}
\]

\[
\text{Det}(A[\bar{I}, \bar{J}]).
\]

\[
\text{Det}(A[I, J]).
\]

\[
\text{Det}(A) = \sum_{I \subseteq [n], |I| = |J|} \text{Det}(A[\bar{I}, \bar{J}]) \cdot \text{Det}(A[I, J]) \cdot (-1)^{|I|+|J|}
\]
Fix a set of columns, $J \subseteq [6]$.

Iterate over all $I \subseteq [6]$ such that $|I| = |J|$.

$$\det(A) = \sum_{I \subseteq [n], |I| = |J|} \det(A[I, \bar{J}]) \cdot \det(A[I, J]) \cdot (-1)^{\sum I + \sum J}$$
Fix a set of columns, $J \subseteq [6]$.

Iterate over all $I \subseteq [6]$ such that $|I| = |J|$.

\[
\begin{vmatrix}
    a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\
    a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\
    a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\
    a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\
    a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\
    a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66}
\end{vmatrix}
\]

\[
\begin{vmatrix}
    a_{12} & a_{14} & a_{15} \\
    a_{21} & a_{23} & a_{26} \\
    a_{31} & a_{33} & a_{36} \\
    a_{41} & a_{44} & a_{45} \\
    a_{51} & a_{53} & a_{56} \\
    a_{62} & a_{64} & a_{65}
\end{vmatrix}
\]

\[
\begin{vmatrix}
    a_{21} & a_{23} & a_{26} \\
    a_{31} & a_{33} & a_{36} \\
    a_{51} & a_{53} & a_{56}
\end{vmatrix}
\]

\[
(-1)^{(1+3+6)+(2+3+5)}
\]

\[
\text{Det}(A[I, J])
\]

\[
\text{Det}(A[I, J])
\]

\[
\text{Det}(A) = \sum_{I \subseteq [n], |I| = |J|} \text{Det}(A[I, J]) \cdot \text{Det}(A[I, J]) \cdot (-1)^{\sum I + \sum J}
\]
Fix a set of columns, $J \subseteq \{6\}$. Iterate over all $I \subseteq \{6\}$ such that $|I| = |J|$.

$$
\begin{vmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\
  a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\
  a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\
  a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\
  a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\
  a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66}
\end{vmatrix}
$$

$$
\begin{vmatrix}
  a_{12} & a_{14} & a_{15} \\
  a_{22} & a_{24} & a_{25} \\
  a_{32} & a_{34} & a_{35} \\
  a_{42} & a_{44} & a_{45} \\
  a_{52} & a_{54} & a_{55} \\
  a_{62} & a_{64} & a_{65}
\end{vmatrix} = (-1)^{(1+3+6)+(2+3+6)}
$$

$$
\det(A_{[I,J]}) \cdot \det(A_{[I,J]}) \cdot (-1)^{\sum I + \sum J}
$$

$$
\det(A) = \sum_{I \subseteq [n], |I| = |J|} \det(A_{[I,J]}) \cdot \det(A_{[I,J]}) \cdot (-1)^{\sum I + \sum J}
$$
Fix a set of columns, $J \subseteq [6]$.

Iterate over all $I \subseteq [6]$ such that $|I| = |J|$.

$$
\begin{vmatrix}
\begin{array}{cccc}
  a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\
  a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\
  a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\
  a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\
  a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\
  a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66}
\end{array}
\end{vmatrix}
$$

$\det(A[I, J])$.

$$
\begin{vmatrix}
\begin{array}{ccc}
  a_{12} & a_{14} & a_{15} \\
  a_{32} & a_{34} & a_{35} \\
  a_{62} & a_{64} & a_{65}
\end{array}
\end{vmatrix}
$$

$\det(A[I])$.

$$
\begin{vmatrix}
\begin{array}{ccc}
  a_{21} & a_{23} & a_{26} \\
  a_{41} & a_{43} & a_{46} \\
  a_{51} & a_{53} & a_{56}
\end{array}
\end{vmatrix}
$$

$(-1)^{(1+3+6)+(2+4+5)}$

$$
\det(A) = \sum_{I \subseteq [n], |I| = |J|} \det(A[I, J]) \cdot \det(A[I, J]) \cdot (-1)^{|I|+|J|}
$$
Fix a set of columns, \( J \subseteq [6] \).

Iterate over all \( I \subseteq [6] \) such that \(|I| = |J|\).

\[
\begin{vmatrix}
\mathbf{a}_{11} & \mathbf{a}_{12} & \mathbf{a}_{13} & \mathbf{a}_{14} & \mathbf{a}_{15} & \mathbf{a}_{16} \\
\mathbf{a}_{21} & \mathbf{a}_{22} & \mathbf{a}_{23} & \mathbf{a}_{24} & \mathbf{a}_{25} & \mathbf{a}_{26} \\
\mathbf{a}_{31} & \mathbf{a}_{32} & \mathbf{a}_{33} & \mathbf{a}_{34} & \mathbf{a}_{35} & \mathbf{a}_{36} \\
\mathbf{a}_{41} & \mathbf{a}_{42} & \mathbf{a}_{43} & \mathbf{a}_{44} & \mathbf{a}_{45} & \mathbf{a}_{46} \\
\mathbf{a}_{51} & \mathbf{a}_{52} & \mathbf{a}_{53} & \mathbf{a}_{54} & \mathbf{a}_{55} & \mathbf{a}_{56} \\
\mathbf{a}_{61} & \mathbf{a}_{62} & \mathbf{a}_{63} & \mathbf{a}_{64} & \mathbf{a}_{65} & \mathbf{a}_{66}
\end{vmatrix}
\]

\[
\begin{vmatrix}
\mathbf{a}_{12} & \mathbf{a}_{14} & \mathbf{a}_{15} \\
\mathbf{a}_{32} & \mathbf{a}_{34} & \mathbf{a}_{35} \\
\mathbf{a}_{52} & \mathbf{a}_{54} & \mathbf{a}_{55}
\end{vmatrix}
\begin{vmatrix}
\mathbf{a}_{21} & \mathbf{a}_{23} & \mathbf{a}_{26} \\
\mathbf{a}_{41} & \mathbf{a}_{43} & \mathbf{a}_{46} \\
\mathbf{a}_{61} & \mathbf{a}_{63} & \mathbf{a}_{66}
\end{vmatrix}
\]

\[
(-1)^{(1+3+6)+(2+4+6)}
\]

\[
\text{Det}(\mathbf{A}[\bar{I}, \bar{J}]).
\]

\[
\text{Det}(\mathbf{A}[I, J]).
\]

\[
\text{Det}(\mathbf{A}) = \sum_{I \subseteq [n], |I| = |J|} \text{Det}(\mathbf{A}[\bar{I}, \bar{J}]) \cdot \text{Det}(\mathbf{A}[I, J]) \cdot (-1)^{\sum I + \sum J}
\]
Fix a set of columns, $J \subseteq [6]$.

Iterate over all $I \subseteq [6]$ such that $|I| = |J|$.

$$\det(A) = \sum_{I \subseteq [n], |I| = |J|} \det(A[I, \overline{J}]) \cdot \det(A[I, J]) \cdot (-1)^{\sum I + \sum J}$$
Fix a set of columns, $J \subseteq [6]$.

Iterate over all $I \subseteq [6]$ such that $|I| = |J|$.

$$\det(A) = \sum_{I \subseteq [n], |I| = |J|} \det(A[I, J]) \cdot \det(A[I, J]) \cdot (-1)^{\sum I + \sum J}$$
Fix a set of columns, $J \subseteq [6]$.

Iterate over all $I \subseteq [6]$ such that $|I| = |J|$.

$$\det(A[I, J]) = \sum_{I \subseteq [n], |I| = |J|} \det(A[I, \bar{J}]) \cdot \det(A[I, J]) \cdot (-1)^{|I| + |J|}$$
Fix a set of columns, $J \subseteq [6]$.

Iterate over all $I \subseteq [6]$ such that $|I| = |J|$.

$$\text{Det}(A[I, J]).$$

$$\text{Det}(A[I, J]).$$

$$\text{Det}(A) = \sum_{I \subseteq [n], |I| = |J|} \text{Det}(A[I, \bar{J}]) \cdot \text{Det}(A[I, J]) \cdot (-1)^{\sum I + \sum J}$$
Fix a set of columns, \( J \subseteq [6] \).
Iterate over all \( I \subseteq [6] \) such that \( |I| = |J| \).

\[
\begin{vmatrix}
\begin{array}{cccccc}
\alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} \\
\alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} & \alpha_{25} & \alpha_{26} \\
\alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} & \alpha_{35} & \alpha_{36} \\
\alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} & \alpha_{45} & \alpha_{46} \\
\alpha_{51} & \alpha_{52} & \alpha_{53} & \alpha_{54} & \alpha_{55} & \alpha_{56} \\
\alpha_{61} & \alpha_{62} & \alpha_{63} & \alpha_{64} & \alpha_{65} & \alpha_{66}
\end{array}
\end{vmatrix}
\]

\[
\begin{vmatrix}
\begin{array}{cccc}
\alpha_{12} & \alpha_{14} & \alpha_{15} \\
\alpha_{22} & \alpha_{24} & \alpha_{25} \\
\alpha_{32} & \alpha_{34} & \alpha_{35}
\end{array}
\end{vmatrix}
\begin{vmatrix}
\begin{array}{cccc}
\alpha_{41} & \alpha_{43} & \alpha_{46} \\
\alpha_{51} & \alpha_{53} & \alpha_{56} \\
\alpha_{61} & \alpha_{63} & \alpha_{66}
\end{array}
\end{vmatrix}
\]

\[
(-1)^{(1+3+6)+(4+5+6)}
\]

\[
\text{Det}(A[\bar{I}, \bar{J}]).
\]

\[
\text{Det}(A[I, J]).
\]

\[
\text{Det}(A) = \sum_{I \subseteq [n], |I| = |J|} \text{Det}(A[\bar{I}, \bar{J}]) \cdot \text{Det}(A[I, J]) \cdot (-1)^{|I|+|J|}
\]
Recall: A Linear (or Representable) Matroid
\( \mathcal{M} = (E, J) \), where \( E = \{ e_1, \ldots, e_n \} \) and \( J \subseteq 2^E \)
\[ \mathcal{M} = (E, J), \text{ where } E = \{e_1, \ldots, e_n\} \text{ and } J \subseteq 2^E \]

Columns indexed by elements of \( E \)

\[ A_M = \begin{pmatrix} \end{pmatrix} \]
$M = (E, J)$, where $E = \{e_1, \ldots, e_n\}$ and $J \subseteq 2^E$

Columns indexed by elements of $E$

\[
A_M = \begin{pmatrix}
\vdots \\
x_{e_1} \\
\vdots
\end{pmatrix}
\]
$\mathcal{M} = (E, J)$, where $E = \{e_1, \ldots, e_n\}$ and $J \subseteq 2^E$

Columns indexed by elements of $E$

$A_M = \begin{pmatrix} x_{e_1} & x_{e_2} \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{pmatrix}$
$M = (E, I)$, where $E = \{e_1, \ldots, e_n\}$ and $I \subseteq 2^E$

Columns indexed by elements of $E$

$$A_M = \begin{pmatrix} x_{e_1} & x_{e_2} & \cdots \end{pmatrix}$$
\[ \mathcal{M} = (E, J), \] where \( E = \{e_1, \ldots, e_n\} \) and \( J \subseteq 2^E \)

Columns indexed by elements of \( E \)

\[
A_{\mathcal{M}} = \begin{pmatrix}
\chi_{e_1} & \chi_{e_2} & \cdots & \chi_{e_i}
\end{pmatrix}
\]
\[ M = (E, J), \text{ where } E = \{e_1, \ldots, e_n\} \text{ and } J \subseteq 2^E \]

Columns indexed by elements of \( E \)

\[ A_M = \begin{pmatrix} x_{e_1} & x_{e_2} & \cdots & x_{e_i} & \cdots \end{pmatrix} \]
\( \mathcal{M} = (E, J), \) where \( E = \{e_1, \ldots, e_n\} \) and \( J \subseteq 2^E \)

Columns indexed by elements of \( E \)

\[
A_M = \begin{pmatrix}
x_{e_1} & x_{e_2} & \cdots & x_{e_i} & \cdots & x_{e_{n-1}}
\end{pmatrix}
\]
\( M = (E, J) \), where \( E = \{e_1, \ldots, e_n\} \) and \( J \subseteq 2^E \)

Columns indexed by elements of \( E \)

\[
A_M = \begin{pmatrix}
    x_{e_1} & x_{e_2} & \cdots & x_{e_i} & \cdots & x_{e_{n-1}} & x_{e_n}
\end{pmatrix}
\]
$\mathcal{M} = (E, I)$, where $E = \{e_1, \ldots, e_n\}$ and $I \subseteq 2^E$

Columns corresponding to $S \in I$

$A_M = \begin{pmatrix}
  x_{e_1} & x_{e_2} & \cdots & x_{e_i} & \cdots & x_{e_{n-1}} & x_{e_n}
\end{pmatrix}$
$\mathcal{M} = (E, I)$, where $E = \{e_1, \ldots, e_n\}$ and $I \subseteq 2^E$

Columns corresponding to $S \in I$

$$A_M = \begin{pmatrix}
\chi_{e_1} & \chi_{e_2} & \cdots & \chi_{e_i} & \cdots & \chi_{e_{n-1}} & \chi_{e_n}
\end{pmatrix}$$

...are linearly independent.
$\mathcal{M} = (E, J)$, where $E = \{e_1, \ldots, e_n\}$ and $J \subseteq 2^E$

Columns that are linearly independent...

\[
A_M = \begin{pmatrix}
\chi_{e_1} & \chi_{e_2} & \cdots & \chi_{e_i} & \cdots & \chi_{e_{n-1}} & \chi_{e_n}
\end{pmatrix}
\]
\[ M = (E, J), \text{ where } E = \{e_1, \ldots, e_n\} \text{ and } J \subseteq 2^E \]

Columns that are linearly independent...

\[
A_M = \begin{pmatrix}
\chi_{e_1} & \chi_{e_2} & \cdots & \chi_{e_i} & \cdots & \chi_{e_{n-1}} & \chi_{e_n} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
\end{pmatrix}
\]

...correspond to sets in \( J \).
$\mathcal{M} = (E, J)$, where $E = \{e_1, \ldots, e_n\}$ and $J \subseteq 2^E$

Columns indexed by elements of $E$

\[
A_M = \begin{pmatrix}
    x_{e_1} & x_{e_2} & \cdots & x_{e_i} & \cdots & x_{e_{n-1}} & x_{e_n}
\end{pmatrix}
\]

$\text{rk}(M)$
Given: A collection of $p$-sized independent sets$^1$:

$$S = \{S_1, \ldots, S_t\}.$$ 

$^1$The rank of the underlying matroid is $(p + q)$. 
Given: A collection of $p$-sized independent sets\(^1\):

$$S = \{S_1, \ldots, S_t\}.$$ 

Want: A $q$-representative subfamily $\hat{S}$ of size $\leq \binom{p+q}{p}$. 

\(^1\)The rank of the underlying matroid is $(p + q)$. 
\[ Z \in S \]
Given \( Z \in S \)

\( Y \subseteq E \)
Given $Z \in S \in J$ and $Y \subseteq E$. 
\[ Z \in S \]
\[ Y \subseteq E \]
\[ \hat{Z} \in \hat{S} \]

\[ \in J \]
Given \( Z \in S \)

\[ \exists j \in J \]

\( Y \subseteq E \)

\[ \exists j \in J \]

\( \hat{Z} \in \hat{S} \)
Given

\[ Z \in \mathcal{S} \]

\[ Y \subseteq \mathcal{E} \]

\[ \hat{Z} \in \hat{\mathcal{S}} \]
Given 

\[ Z \in \mathcal{S} \]

\[ Y \subseteq \mathcal{E} \]

\[ \hat{Z} \in \hat{\mathcal{S}} \]
Given

\[ |Z| = p \]

\[ Y \subseteq E \]

\[ \hat{Z} \in \hat{S} \]
Given

\|Z\| = p \\
\|\hat{Y}\| = q \\
\hat{Z} \in \hat{S} \\
\in J

Given

\[ |Z| = p \]

\[ |Y| = q \]

\[ |\hat{Z}| = p \]
$A_M = \begin{pmatrix} Z \end{pmatrix} \begin{pmatrix} I \end{pmatrix} (p + q)$
$A_M = \begin{pmatrix} \vdots \\ \vdots \end{pmatrix}$

Columns corresponding to $Z$

$p$

$\text{Det}\left( Y_0 - Y^j_p(A_I; Y) \right) \langle p + q \rangle$
\[ A_M = \begin{pmatrix} \text{Columns corresponding to } Z & \text{Columns corresponding to } Y \\ \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \]
\[ A_M = \begin{pmatrix} \text{Columns corresponding to } Z & \text{Columns corresponding to } Y \\ \end{pmatrix} \]

\[ \text{Det}(A_M[\star, Z \cup Y]) \]
\[\det(A_M[p, \star, Z \cup Y]) \neq 0\]
\( 0 \neq \Det(A_M[\star, Z \cup Y]) \)
\[
A_M = \begin{pmatrix}
\text{Columns corresponding to } Z \cup Y: \text{LINEARLY INDEPENDENT}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\text{Columns corresponding to } Z \\
\text{Columns corresponding to } Y
\end{pmatrix}
\]

\[
0 \neq \text{Det}(A_M[\star, Z \cup Y]) = \sum_{I \subseteq [p+q], |I|=p} \text{Det}(A[I, Z]) \cdot \text{Det}(A[\bar{I}, Y]) \cdot \emptyset
\]
\[0 \neq \text{Det}(A_M[\star, Z \cup Y]) = \sum_{I \subseteq [p+q], |I| = p} \text{Det}(A[I, Z]) \cdot \text{Det}(A[I, Y]) \cdot \emptyset\]
\[ 0 \neq \text{Det}(A_M[\star, Z \cup Y]) = \sum_{I \subseteq [p+q], |I| = p} \text{Det}(A[I, Z]) \cdot \text{Det}(A[\bar{I}, Y]) \cdot \emptyset \]
$0 \neq \det(A_M[\star, Z \cup Y]) = \sum_{I \subseteq [p+q], |I|=p} \det(A[I, Z]) \cdot \det(A[\bar{I}, Y]) \cdot \emptyset$

$v_z := \left( \det(A[I_0, Z]), \ldots, \det(A[I_j, Z]), \ldots, \det(A[I_r, Z]) \right)$
\[ 0 \neq \text{Det}(A_M[\star, Z \cup Y]) = \sum_{I \subseteq [p+q], |I| = p} \text{Det}(A[I, Z]) \cdot \text{Det}(A[I, Y]) \cdot \emptyset \]

\[ v_Z := \left( \begin{array}{c}
\text{Det}(A[I_0, Z]), \ldots, \text{Det}(A[I_j, Z]), \ldots, \text{Det}(A[I_r, Z])
\end{array} \right) \]

All subsets of size \( p \) of \((p + q)\).
\[ 0 \neq \text{Det}(A_M[\star, Z \cup Y]) = \sum_{I \subseteq [p+q], |I| = p} \text{Det}(A[I, Z]) \cdot \text{Det}(A[\bar{I}, Y]) \cdot \emptyset \]

\[ v_Z := \left( \begin{array}{cccc}
\text{Det}(A[I_0, Z]), & \ldots, & \text{Det}(A[I_j, Z]), & \ldots, & \text{Det}(A[I_r, Z])
\end{array} \right) \]

All subsets of size \( p \) of \( (p + q) \).

\[ S = \{S_1, \ldots, S_i, \ldots, S_t\} \]
\[ 0 \neq \text{Det}(A_M[\star, Z \cup Y]) = \sum_{I \subseteq [p+q], |I| = p} \text{Det}(A[I, Z]) \cdot \text{Det}(A[\bar{I}, Y]) \cdot \varnothing \]

\[ v_\mathbb{Z} := \left( \begin{array}{c} \text{Det}(A[I_0, Z]) \quad \ldots \quad \text{Det}(A[I_j, Z]) \quad \ldots \quad \text{Det}(A[I_r, Z]) \end{array} \right) \]

All subsets of size \( p \) of \( (p + q) \).

\[ S = \{S_1, \ldots, S_i, \ldots, S_t\} \]

\[ v_S := \left( \begin{array}{c} \text{Det}(A[I_0, S_1]) \quad \ldots \quad \text{Det}(A[I_j, S_1]) \quad \ldots \quad \text{Det}(A[I_r, S_1]) \end{array} \right) \]
\[ 0 \neq \text{Det}(A_M[\star, Z \cup Y]) = \sum_{I \subseteq [p+q], |I| = p} \text{Det}(A[I, Z]) \cdot \text{Det}(A[\overline{I}, Y]) \cdot \emptyset \]

\[ \nu_Z := \left( \begin{array}{c} \text{Det}(A[I_0, Z]) \ , \ldots \ , \ \text{Det}(A[I_j, Z]) \ , \ldots \ , \ \text{Det}(A[I_r, Z]) \end{array} \right) \]

All subsets of size \( p \) of \( (p + q) \).

\[ S = \{ S_1, \ldots, S_i, \ldots, S_t \} \]

\[ \nu_{S_1} := \left( \begin{array}{c} \text{Det}(A[I_0, S_1]) \ , \ldots \ , \ \text{Det}(A[I_j, S_1]) \ , \ldots \ , \ \text{Det}(A[I_r, S_1]) \end{array} \right) \]

\[ \vdots \]
\( 0 \neq \text{Det}(A_M[\star, Z \cup Y]) = \sum_{I \subseteq [p+q], |I|=p} \text{Det}(A[I, Z]) \cdot \text{Det}(A[\bar{I}, Y]) \cdot \emptyset \)

\( v_Z := \begin{pmatrix}
\text{Det}(A[I_0, Z]) , \ldots , \text{Det}(A[I_j, Z]) , \ldots , \text{Det}(A[I_r, Z]) \\
\end{pmatrix} \)

All subsets of size \( p \) of \( (p + q) \).

\[ S = \{S_1, \ldots, S_i, \ldots, S_t\} \]

\( v_{S_1} := \begin{pmatrix}
\text{Det}(A[I_0, S_1]) , \ldots , \text{Det}(A[I_j, S_1]) , \ldots , \text{Det}(A[I_r, S_1]) \\
\end{pmatrix} \)

\( \vdots \)

\( v_{S_t} := \begin{pmatrix}
\text{Det}(A[I_0, S_t]) , \ldots , \text{Det}(A[I_j, S_t]) , \ldots , \text{Det}(A[I_r, S_t]) \\
\end{pmatrix} \)
\[ 0 \neq \det(A_M[\star, Z \cup Y]) = \sum_{I \subseteq [p+q], |I|=p} \det(A[I, Z]) \cdot \det(A[\bar{I}, Y]) \cdot \emptyset \]

\[ v_Z := \left( \begin{array}{ccccc} \det(A[I_0, Z]) & \ldots & \det(A[I_j, Z]) & \ldots & \det(A[I_r, Z]) \\ \end{array} \right) \]

All subsets of size \( p \) of \( (p + q) \).

\[ S = \{S_1, \ldots, S_i, \ldots, S_t\} \]

\[ v_{S_1} := \left( \begin{array}{ccccc} \det(A[I_0, S_1]) & \ldots & \det(A[I_j, S_1]) & \ldots & \det(A[I_r, S_1]) \\ \end{array} \right) \]

\[ \vdots \]

\[ v_{S_t} := \left( \begin{array}{ccccc} \det(A[I_0, S_t]) & \ldots & \det(A[I_j, S_t]) & \ldots & \det(A[I_r, S_t]) \\ \end{array} \right) \]

\[ \vdots \]
\[ 0 \neq \det(\mathcal{A}_M[\star, Z \cup Y]) = \sum_{I \subseteq [p+q], |I|=p} \det(\mathcal{A}[I, Z]) \cdot \det(\mathcal{A}[^I, Y]) \cdot \emptyset \]

\[ \nu_Z := \left( \det(\mathcal{A}[I_0, Z]) , \ldots , \det(\mathcal{A}[I_j, Z]) , \ldots , \det(\mathcal{A}[I_r, Z]) \right) \]

All subsets of size \( p \) of \( (p + q) \).

\[ S = \{S_1, \ldots, S_i, \ldots, S_t\} \]

\[ \nu_{S_i} := \left( \det(\mathcal{A}[I_0, S_i]) , \ldots , \det(\mathcal{A}[I_j, S_i]) , \ldots , \det(\mathcal{A}[I_r, S_i]) \right) \]

\[ \vdots \]

\[ \nu_{S_t} := \left( \det(\mathcal{A}[I_0, S_t]) , \ldots , \det(\mathcal{A}[I_j, S_t]) , \ldots , \det(\mathcal{A}[I_r, S_t]) \right) \]
0 \neq \det(A_{M[\star, Z \cup Y]}) = \sum_{I \subseteq [p+q], |I|=p} \det(A[I, Z]) \cdot \det(A[\bar{I}, Y]) \cdot \emptyset

\nu_Z := \left( \begin{array}{c} \det(A[I_0, Z]) \vspace{1em} \vdots \vspace{1em} \det(A[I_j, Z]) \vspace{1em} \vdots \vspace{1em} \det(A[I_r, Z]) \end{array} \right)

All subsets of size $p$ of $(p+q)$.

\begin{align*}
\mathcal{S} &= \{S_1, \ldots, S_i, \ldots, S_t\} \\
\nu_{S_1} := &\left( \begin{array}{c} \det(A[I_0, S_1]) \vspace{1em} \vdots \vspace{1em} \det(A[I_j, S_1]) \vspace{1em} \vdots \vspace{1em} \det(A[I_r, S_1]) \end{array} \right) \\
\vdots &
\end{align*}

\begin{align*}
\nu_{S_t} := &\left( \begin{array}{c} \det(A[I_0, S_t]) \vspace{1em} \vdots \vspace{1em} \det(A[I_j, S_t]) \vspace{1em} \vdots \vspace{1em} \det(A[I_r, S_t]) \end{array} \right) \\
\vdots &
\end{align*}

\chi(S) := \{\nu_{S_1}, \ldots, \nu_{S_i}, \ldots, \nu_{S_t}\}
\[ 0 \neq \text{Det}(A_M[\star, Z \cup Y]) = \sum_{I \subseteq [p+q], |I| = p} \text{Det}(A[I, Z]) \cdot \text{Det}(A[\bar{I}, Y]) \cdot \emptyset \]

\[ v_Z := \left( \begin{array}{c}
\text{Det}(A[I_0, Z]) \\
\vdots \\
\text{Det}(A[I_r, Z])
\end{array} \right) \]

All subsets of size \( p \) of \( (p + q) \).

\[
S = \{S_1, \ldots, S_i, \ldots, S_t\}
\]

\[ v_{S_1} := \left( \begin{array}{c}
\text{Det}(A[I_0, S_1]) \\
\vdots \\
\text{Det}(A[I_r, S_1])
\end{array} \right) \]

\[ v_{S_t} := \left( \begin{array}{c}
\text{Det}(A[I_0, S_t]) \\
\vdots \\
\text{Det}(A[I_r, S_t])
\end{array} \right) \]

\[ \chi(S) := \{v_{S_1}, \ldots, v_{S_i}, \ldots, v_{S_t}\} \]
\[ 0 \neq \text{Det}(A_M[\star, Z \cup Y]) = \sum_{I \subseteq [p+q], |I| = p} \text{Det}(A[I, Z]) \cdot \text{Det}(A[I, Y]) \cdot \emptyset \]

\[ \nu_Z := \left( \begin{array}{cccc} \text{Det}(A[I_0, Z]) & \cdots & \text{Det}(A[I_j, Z]) & \cdots & \text{Det}(A[I_r, Z]) \\ \end{array} \right) \]

All subsets of size \( p \) of \((p + q)\).

\[ S = \{S_1, \ldots, S_i, \ldots, S_t\} \]

\[ \nu_{T_1} := \left( \begin{array}{cccc} \text{Det}(A[I_0, T_1]) & \cdots & \text{Det}(A[I_j, T_1]) & \cdots & \text{Det}(A[I_r, T_1]) \\ \end{array} \right) \]

\[ \vdots \]

\[ \nu_{T_r} := \left( \begin{array}{cccc} \text{Det}(A[I_0, T_r]) & \cdots & \text{Det}(A[I_j, T_r]) & \cdots & \text{Det}(A[I_r, T_r]) \\ \end{array} \right) \]

A basis of size \( \leq \binom{p+q}{p} \) for

\[ \chi(S) := \{\nu_{S_1}, \ldots, \nu_{S_i}, \ldots, \nu_{S_t}\} \]
\[ 0 \neq \det(A_M[\star, Z \cup Y]) = \sum_{I \subseteq [p+q], |I|=p} \det(A[I, Z]) \cdot \det(A[\bar{I}, Y]) \cdot \emptyset \]

\[ v_Z := \left( \det(A[I_0, Z]), \ldots, \det(A[I_j, Z]), \ldots, \det(A[I_r, Z]) \right) \]

\[ v_Z = \lambda_1 v_{T_1} + \cdots + \lambda_r v_{T_r} \]

\[ S = \{S_1, \ldots, S_i, \ldots, S_t\} \]

\[ v_{T_1} := \left( \det(A[I_0, T_1]), \ldots, \det(A[I_j, T_1]), \ldots, \det(A[I_r, T_1]) \right) \]

\[ \vdots \]

\[ v_{T_r} := \left( \det(A[I_0, T_r]), \ldots, \det(A[I_j, T_r]), \ldots, \det(A[I_r, T_r]) \right) \]

A basis of size \( \leq (p^+q) \) for

\[ \chi(S) := \{v_{S_1}, \ldots, v_{S_i}, \ldots, v_{S_t}\} \]
\[ 0 \neq \text{Det}(A_M[\star, Z \cup Y]) = \sum_{I \subseteq [p+q], |I| = p} \text{Det}(A[I, Z]) \cdot \text{Det}(A[\bar{I}, Y]) \cdot \emptyset \]

\[ v_Z := \left( \text{Det}(A[I_0, Z]), \ldots, \text{Det}(A[I_J, Z]), \ldots, \text{Det}(A[I_r, Z]) \right) \]

\[ v_Z = \lambda_1 v_{T_1} + \cdots + \lambda_r v_{T_r} \]

\[ S = \{S_1, \ldots, S_i, \ldots, S_t\} \]

\[ v_{T_1} := \left( \text{Det}(A[I_0, T_1]), \ldots, \text{Det}(A[I_J, T_1]), \ldots, \text{Det}(A[I_r, T_1]) \right) \]

\[ v_{T_r} := \left( \text{Det}(A[I_0, T_r]), \ldots, \text{Det}(A[I_J, T_r]), \ldots, \text{Det}(A[I_r, T_r]) \right) \]

A basis of size \( \leq \binom{p+q}{p} \) for

\[ \chi(S) = \{v_{S_1}, \ldots, v_{S_i}, \ldots, v_{S_t}\} \]
\[ 0 \neq \det(A_M[\star, Z \cup Y]) = \sum_{I \subseteq [p+q], |I| = p} \det(A[I, Z]) \cdot \det(A[\bar{I}, Y]) \cdot \emptyset \]

\[ \nu_Z := \left( \det(A[I_0, Z]) , \ldots , \det(A[I_j, Z]) , \ldots , \det(A[I_r, Z]) \right) \]

\[ \nu_Z = \lambda_1 \nu_{T_1} + \cdots + \lambda_r \nu_{T_r} \]

\[ \nu_Z[I] = \lambda_1 \nu_{T_1}[I] + \cdots + \lambda_r \nu_{T_r}[I] \]

\[ \nu_{T_1} := \left( \det(A[I_0, T_1]) , \ldots , \det(A[I_j, T_1]) , \ldots , \det(A[I_r, T_1]) \right) \]

\[ \vdots \]

\[ \nu_{T_r} := \left( \det(A[I_0, T_r]) , \ldots , \det(A[I_j, T_r]) , \ldots , \det(A[I_r, T_r]) \right) \]

A basis of size \( \leq \binom{p+q}{p} \) for

\[ \chi(S) := \{v_{S_1}, \ldots, v_{S_i}, \ldots, v_{S_t}\} \]
\[ 0 \neq \text{Det}(A_{M[\star, Z \cup Y]}) = \sum_{I \subseteq [p+q], |I|=p} \text{Det}(A[I, Z]) \cdot \text{Det}(A[I, Y]) \cdot \emptyset \]

\[ \nu_Z := \begin{pmatrix} \text{Det}(A[I_0, Z]) & \ldots & \text{Det}(A[I_j, Z]) & \ldots & \text{Det}(A[I_r, Z]) \end{pmatrix} \]

\[ \nu_Z = \lambda_1 \nu_{T_1} + \cdots + \lambda_r \nu_{T_r} \]

\[ \nu_Z[I] = \lambda_1 \nu_{T_1}[I] + \cdots + \lambda_r \nu_{T_r}[I] \]

\[ \text{Det}(A[I, Z]) = \lambda_1 \text{Det}(A[I, T_1]) + \cdots + \lambda_r \text{Det}(A[I, T_r]) \]

\[ \nu_{T_1} := \begin{pmatrix} \text{Det}(A[I_0, T_1]) & \ldots & \text{Det}(A[I_j, T_1]) & \ldots & \text{Det}(A[I_r, T_1]) \end{pmatrix} \]

\[ \vdots \]

\[ \nu_{T_r} := \begin{pmatrix} \text{Det}(A[I_0, T_r]) & \ldots & \text{Det}(A[I_j, T_r]) & \ldots & \text{Det}(A[I_r, T_r]) \end{pmatrix} \]

A basis of size \( \leq \binom{p+q}{p} \) for

\[ \chi(S) := \{\nu_{S_1}, \ldots, \nu_{S_i}, \ldots, \nu_{S_t}\} \]
$0 \neq \text{Det}(A_M[\star, Z \cup Y]) = \sum_{I \subseteq \{p+q\}, |I| = p} \sum_{i=1}^{r} \lambda_i \text{Det}(A[I, T_i]) \cdot \text{Det}(A[\bar{I}, Y]) \cdot \varnothing$

$v_Z := \left( \text{Det}(A[I_0, Z]), \ldots, \text{Det}(A[I_j, Z]), \ldots, \text{Det}(A[I_r, Z]) \right)$

$v_Z = \lambda_1 v_{T_1} + \cdots + \lambda_r v_{T_r}$

$v_Z[I] = \lambda_1 v_{T_1}[I] + \cdots + \lambda_r v_{T_r}[I]$

$\text{Det}(A[I, Z]) = \lambda_1 \text{Det}(A[I, T_1]) + \cdots + \lambda_r \text{Det}(A[I, T_r])$

$v_{T_1} := \left( \text{Det}(A[I_0, T_1]), \ldots, \text{Det}(A[I_j, T_1]), \ldots, \text{Det}(A[I_r, T_1]) \right)$

$\vdots$

$v_{T_r} := \left( \text{Det}(A[I_0, T_r]), \ldots, \text{Det}(A[I_j, T_r]), \ldots, \text{Det}(A[I_r, T_r]) \right)$

A basis of size $\leq \binom{p+q}{p}$ for

$\chi(S) := \{v_{S_1}, \ldots, v_{S_i}, \ldots, v_{S_t}\}$
\[
0 \neq \det(A_M[* , Z \cup Y]) = \sum_{i=1}^{r} \sum_{I \subseteq [p+q], |I|=p} \det(A[I, T_i]) \cdot \det(A[\overline{I}, Y]) \cdot \emptyset
\]

\[\nu_Z := \left( \det(A[I_0, Z]) , \ldots , \det(A[I_j, Z]) , \ldots , \det(A[I_r, Z]) \right)\]

\[\nu_Z = \lambda_1 \nu_{T_1} + \cdots + \lambda_r \nu_{T_r}\]

\[\nu_Z[I] = \lambda_1 \nu_{T_1}[I] + \cdots + \lambda_r \nu_{T_r}[I]\]

\[\det(A[I, Z]) = \lambda_1 \det(A[I, T_1]) + \cdots + \lambda_r \det(A[I, T_r])\]

\[\nu_{T_1} := \left( \det(A[I_0, T_1]) , \ldots , \det(A[I_j, T_1]) , \ldots , \det(A[I_r, T_1]) \right)\]

\[\vdots\]

\[\nu_{T_r} := \left( \det(A[I_0, T_r]) , \ldots , \det(A[I_j, T_r]) , \ldots , \det(A[I_r, T_r]) \right)\]

A basis of size \(\leq \binom{p+q}{p}\) for

\[\chi(S) := \{\nu_{S_1}, \ldots, \nu_{S_l}, \ldots, \nu_{S_t}\}\]
\[ 0 \neq \text{Det}(A_M[\star, Z \cup Y]) = \sum_{i=1}^{r} \text{Det}(A_M[\star, T_i \cup Y]) \]

\[ v_Z := \left( \text{Det}(A[I_0, Z]), \ldots, \text{Det}(A[I_j, Z]), \ldots, \text{Det}(A[I_r, Z]) \right) \]

\[ v_Z = \lambda_1 v_{T_1} + \cdots + \lambda_r v_{T_r} \]

\[ v_Z[I] = \lambda_1 v_{T_1}[I] + \cdots + \lambda_r v_{T_r}[I] \]

\[ \text{Det}(A[I, Z]) = \lambda_1 \text{Det}(A[I, T_1]) + \cdots + \lambda_r \text{Det}(A[I, T_r]) \]

\[ v_{T_1} := \left( \text{Det}(A[I_0, T_1]), \ldots, \text{Det}(A[I_j, T_1]), \ldots, \text{Det}(A[I_r, T_1]) \right) \]

\[ \vdots \]

\[ v_{T_r} := \left( \text{Det}(A[I_0, T_r]), \ldots, \text{Det}(A[I_j, T_r]), \ldots, \text{Det}(A[I_r, T_r]) \right) \]

A basis of size \( \leq \binom{p+q}{p} \) for

\[ \chi(S) := \{v_{S_1}, \ldots, v_{S_i}, \ldots, v_{S_t}\} \]
\[
0 \neq \text{Det}(A_M[\star, Z \cup Y]) = \sum_{i=1}^{r} \text{Det}(A_M[\star, T_i \cup Y])
\]

\[
v_Z := \begin{pmatrix} \text{Det}(A[I_0, Z]) & \ldots & \text{Det}(A[I_j, Z]) & \ldots & \text{Det}(A[I_r, Z]) \end{pmatrix}
\]

\[
v_Z = \lambda_1 v_{T_1} + \cdots + \lambda_r v_{T_r}
\]

\[
v_Z[I] = \lambda_1 v_{T_1}[I] + \cdots + \lambda_r v_{T_r}[I]
\]

\[
\text{Det}(A[I, Z]) = \lambda_1 \text{Det}(A[I, T_1]) + \cdots + \lambda_r \text{Det}(A[I, T_r])
\]

\[
v_{T_1} := \begin{pmatrix} \text{Det}(A[I_0, T_1]) & \ldots & \text{Det}(A[I_j, T_1]) & \ldots & \text{Det}(A[I_r, T_1]) \end{pmatrix}
\]

\[
\vdots
\]

\[
v_{T_r} := \begin{pmatrix} \text{Det}(A[I_0, T_r]) & \ldots & \text{Det}(A[I_j, T_r]) & \ldots & \text{Det}(A[I_r, T_r]) \end{pmatrix}
\]

A basis of size \( \leq \binom{p+q}{p} \) for

\[
\chi(S) := \{v_{S_1}, \ldots, v_{S_i}, \ldots, v_{S_t}\}
\]
\[ 0 \neq \text{Det}(\mathcal{A}_M[\star, Z \cup Y]) = \sum_{i=1}^{r} \text{Det}(\mathcal{A}_M[\star, T_i \cup Y]) \]

Note that at for at least one \( T_i \), we have that:
\[
\text{Det}(\mathcal{A}_M[\star, T_i \cup Y]) \neq 0
\]
For such a $T_i$, we know that:

1. $Y \cap T_i = \emptyset$ (easily checked: all terms that survive have this property),
2. $Y \cup T_i \in J$ (since non-zero determinant $\rightarrow$ linearly independent columns).
0 \neq \text{Det}(A_{M}[\star, Z \cup Y]) = \sum_{i=1}^{r} \text{Det}(A_{M}[\star, T_i \cup Y])

Note that for at least one $T_i$, we have that:
\[
\text{Det}(A_{M}[\star, T_i \cup Y]) \neq 0
\]

For such a $T_i$, we know that:
\begin{enumerate}
  \item $Y \cap T_i = \emptyset$ (easily checked: all terms that survive have this property),
  \item $Y \cup T_i \in \mathcal{I}$ (since non-zero determinant $\rightarrow$ linearly independent columns).
\end{enumerate}

Thus, the sets corresponding to the basis vectors, $T_1, \ldots, T_r$, do form a $q$-representative family.
\[ 0 \neq \text{Det}(A_M[\star, Z \cup Y]) = \sum_{i=1}^{r} \text{Det}(A_M[\star, T_i \cup Y]) \]

\[ v_Z := \left( \text{Det}(A[I_0, Z]), \ldots, \text{Det}(A[I_j, Z]), \ldots, \text{Det}(A[I_r, Z]) \right) \]

\[ v_Z = \lambda_1 v_{T_1} + \cdots + \lambda_r v_{T_r} \]

\[ v_Z[I] = \lambda_1 v_{T_1}[I] + \cdots + \lambda_r v_{T_r}[I] \]

\[ \text{Det}(A[I, Z]) = \lambda_1 \text{Det}(A[I, T_1]) + \cdots + \lambda_r \text{Det}(A[I, T_r]) \]

\[ v_{T_1} := \left( \text{Det}(A[I_0, T_1]), \ldots, \text{Det}(A[I_j, T_1]), \ldots, \text{Det}(A[I_r, T_1]) \right) \]

\[ \vdots \]

\[ v_{T_r} := \left( \text{Det}(A[I_0, T_r]), \ldots, \text{Det}(A[I_j, T_r]), \ldots, \text{Det}(A[I_r, T_r]) \right) \]

A basis of size \( \binom{p+q}{p} \) for

\[ \chi(S) := \{ v_{S_1}, \ldots, v_{S_t}, \ldots, v_{S_t} \} \]
Computing $T_1, \ldots, T_r$. 

We form a matrix with the vectors $\{v_{S_1}, \ldots, v_{S_i}, \ldots, v_{S_t}\}$ as the columns:
Computing $T_1, \ldots, T_r$.

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Computing $T_1, \ldots, T_r$.

We form a matrix with the vectors $\{v_{S_1}, \ldots, v_{S_i}, \ldots, v_{S_t}\}$ as the columns:

$$
\begin{pmatrix}
\vdots \\
v_{S_1} \\
\vdots \\
\end{pmatrix}
$$
We form a matrix with the vectors \( \{v_{S_1}, \ldots, v_{S_i}, \ldots, v_{S_t}\} \) as the columns:
Computing $T_1, \ldots, T_r$.

We form a matrix with the vectors $\{v_{S_1}, \ldots, v_{S_i}, \ldots, v_{S_t}\}$ as the columns:
Computing $T_1, \ldots, T_r$. 

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Computing $T_1, \ldots, T_r$.

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We form a matrix with the vectors $\{v_{S_1}, \ldots, v_{S_i}, \ldots, v_{S_t}\}$ as the columns:
Computing $T_1, \ldots, T_r$.

We form a matrix with the vectors $\{v_{S_1}, \ldots, v_{S_i}, \ldots, v_{S_t}\}$ as the columns:

\[
\begin{pmatrix}
    v_{S_1} & v_{S_2} & \cdots & v_{S_i} & \cdots & v_{S_{t-1}} & v_{S_t}
\end{pmatrix}
\]
Computing $T_1, \ldots, T_r$.

We form a matrix with the vectors $\{v_{S_1}, \ldots, v_{S_i}, \ldots, v_{S_t}\}$ as the columns:

\[
\begin{pmatrix}
v_{S_1} & v_{S_2} & \ldots & v_{S_i} & \ldots & v_{S_{t-1}} & v_{S_t}
\end{pmatrix}
\]

...and compute a column basis.
\[
\begin{pmatrix}
[A[I_0, S_1]] & [A[I_0, S_2]] & \ldots & [A[I_0, S_i]] & \ldots & [A[I_0, S_t]] \\
[A[I_1, S_1]] & [A[I_1, S_2]] & \ldots & [A[I_1, S_i]] & \ldots & [A[I_1, S_t]] \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
[A[I_j, S_1]] & [A[I_j, S_2]] & \ldots & [A[I_j, S_i]] & \ldots & [A[I_j, S_t]] \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
[A[I_r, S_1]] & [A[I_r, S_2]] & \ldots & [A[I_r, S_i]] & \ldots & [A[I_r, S_t]]
\end{pmatrix}
\]
\[
\begin{bmatrix}
[A[I_0, S_1]] & [A[I_0, S_2]] & \cdots & [A[I_0, S_i]] & \cdots & [A[I_0, S_t]] \\
[A[I_1, S_1]] & [A[I_1, S_2]] & \cdots & [A[I_1, S_i]] & \cdots & [A[I_1, S_t]] \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
[A[I_j, S_1]] & [A[I_j, S_2]] & \cdots & [A[I_j, S_i]] & \cdots & [A[I_j, S_t]] \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
[A[I_r, S_1]] & [A[I_r, S_2]] & \cdots & [A[I_r, S_i]] & \cdots & [A[I_r, S_t]]
\end{bmatrix}
\]
\[
\begin{pmatrix}
[A[I_0, S_1]] & [A[I_0, S_2]] & \ldots & [A[I_0, S_i]] & \ldots & [A[I_0, S_t]] \\
[A[I_1, S_1]] & [A[I_1, S_2]] & \ldots & [A[I_1, S_i]] & \ldots & [A[I_1, S_t]] \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
[A[I_j, S_1]] & [A[I_j, S_2]] & \ldots & [A[I_j, S_i]] & \ldots & [A[I_j, S_t]] \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
[A[I_r, S_1]] & [A[I_r, S_2]] & \ldots & [A[I_r, S_i]] & \ldots & [A[I_r, S_t]]
\end{pmatrix}
\]

\( (p+q) \) rows

\( t \) columns
t \cdot \left( \begin{array}{c} p + q \\ q \end{array} \right) \text{Determinant Computations.}
Let $\mathcal{M}$ be a linear matroid of rank $p + q = k$, $\mathcal{S} = \{S_1, \ldots, S_t\}$ be a $p$-family of independent sets. Then there exists a $q$-representative of size at most $\binom{p + q}{q}$. Moreover, given a representation of $\mathcal{M}$ over a field $F$, we can find such a representative family in $O\left(\frac{(p + q)^t}{t^p}\right)$ operations over $F$. 
Let $\mathcal{M}$ be a linear matroid of rank $p + q = k$, $S = \{S_1, \ldots, S_t\}$ be a $p$-family of independent sets. Then there exists a $q$-representative of size at most $\binom{p+q}{q}$.

Moreover, given a representation of $\mathcal{M}$ over a field $\mathbb{F}$, we can find such a representative family in $O\left(\binom{p+q}{q} tp^\omega + t\binom{p+q}{q}^\omega - 1\right)$ operations over $\mathbb{F}$. 
REPRESENTATIVE SETS

And that will be all!