Exercise 1 Longest Palindrome Subsequence Problem: Given a sequence $S$ of length $n$, find the longest subsequence which is a palindrome (i.e., a sequence of characters which reads the same backward and forward).

Prove that if this problem can be solved in time $O(n^{2-\varepsilon})$ then OVH fails.

Exercise 2 Diameter Problem: Given a graph $G$ on $n$ vertices and $m$ edges, compute the largest distance between any two vertices in $G$.

We consider sparse graphs, i.e., $m = \tilde{O}(n) = O(n \text{polylog } n)$. Show that the diameter can be computed in time $\tilde{O}(n^2)$, and prove that if the diameter can be computed in time $O(n^{2-\varepsilon})$ then OVH fails.

Exercise 3 $k$-Clique Problem: Given a graph $G$ on $n$ vertices, decide whether there are vertices $v_1, \ldots, v_k$ that are pairwise adjacent.

Show by a reduction that if OVH fails, i.e., OV can be solved in time $O(n^{2-\varepsilon}\text{poly}(d))$, then for sufficiently large $k$ and some $\varepsilon' > 0$ the $k$-Clique problem can be solved in time $O(n^{k-\varepsilon'})$.

Remark: The fastest known algorithm for $k$-Clique runs in time $O(n^{k^{\omega/3}})$ where $\omega \leq 2.37$, so this reduction does not yield a tight lower bound for OV, but is only a partial relation.

Exercise 4 RegExpMatching: Given a regular expression $R$ of size $m$ and a text $T$ of length $n$, determine whether any substring $T'$ of $T$ can be derived from $R$.

It is well-known that this problem can be solved in time $O(nm)$. Show that there is no algorithm running in time $O((mn)^{1-\varepsilon})$ unless OVH fails.

For specific classes of regular expressions there are faster algorithms to solve this problem. Consider homogeneous regular expressions: A regular expression $R$ is called homogeneous of type $o_1o_2\ldots o_\ell$ (where $o_i \in \{\circ, \ast, +, |\}$) if there exist $a_1, \ldots, a_p$, which are characters or homogeneous regular expressions of type $o_2, \ldots, o_\ell$, such that $R = o_1(a_1, \ldots, a_p)$. For example, the regular expression $[(a \circ b \circ c) | b | (a \circ b)]^*$ is homogeneous of type $\ast | \circ$, but the regular expression $(a^*) | (b^+)$ is not homogeneous.

a) Find types $t$ such that RegExpMatching restricted to homogeneous regular expression of type $t$ can be solved in time $O(n + m)$.

b) Prove that there is no $O((mn)^{1-\varepsilon})$ algorithm for RegExpMatching restricted to homogeneous regular expression of type $| \circ |$ unless OVH fails.

Prove the same result for homogeneous regular expressions of type $| \circ \ast |$. 