# Handling Real Arithmetic with Infinite Word Automata

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# An Automata-Theoretic Approach to Real Arithmetic

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# On the Unusual Effectiveness of Computer Science in Logic

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## The Starting Point

- Using finite automata to represent sets of integers is an interesting and potentially practical approach (the LASH tool).
- Extending this representation to reals can be done quite naturally and yields a tool for handling the combined theory of integers and reals.
- Handling the reals is done by moving to automata on infinite words, which from a practical algorithmic point of view is quite problematic.
- This is surprising since the additive theory of the reals is easier to handle than the corresponding theory over the integers.
- Can this be explained ? Yes! *A very special type of infinite word automata are sufficient for handling the additive theory of the reals and integers.*

## Representing sets of Real Vectors by Automata : The Real Vector Automata (RVA)

- Reals are encoded in a base $r > 1$ by infinite words built on the alphabet $\{0, \ldots, r-1, \star\}$. Negative numbers are encoded using $r$'s complement.

  **Examples :**

$$\begin{aligned} L_2(3.5) &= 0^+11\star 1(0)^\omega \cup 0^+11\star 0(1)^\omega \\ L_2(-4) &= 1^+00\star(0)^\omega \cup 1^+011\star(1)^\omega; \end{aligned}$$

- Vectors with $n$ real components are encoded by infinite words over the alphabet $\{0, \ldots, r-1\}^n \cup \{\star\}$.

- An RVA representing a set $S \subseteq \mathbf{R}^n$ is a Büchi automaton accepting all the base $r$ encodings of the vectors in $S$.

## Properties of RVA

- RVAs representing sets of the form $\{\vec{x} \in \mathbf{R}^n \mid \vec{a}.\vec{x} \left\{ \begin{array}{c} = \\ \leq \end{array} \right\} b\}$, with $\vec{a} \in \mathbf{Z}^n, b \in \mathbf{Z}$, can easily be constructed;
- The set $\mathbf{Z}$ is representable by an RVA;
- Given RVAs representing sets $S_1, S_2 \subseteq \mathbf{R}^n$, it is possible to algorithmically construct RVAs representing the sets
  - $S_1 \cup S_2$, $S_1 \cap S_2$, $S_1 \times S_2$,
  - $\overline{S_1} = \mathbf{R}^n \setminus S_1$,
  - $S_1|_{\neq i} = \{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \mid (\exists x_i \in \mathbf{R})((x_1, \ldots, x_n) \in S_1)\}$;
- It is decidable whether the set represented by an RVA is empty or not.

## RVAs and arithmetic

It follows from the properties above that, for every subset of $\mathbf{R}^n$ definable in the first-order theory of $\langle \mathbf{R}, \mathbf{Z}, +, \leq \rangle$, one can algorithmically construct an RVA that represents it.

RVAs can thus be used as a tool to decide this theory.

**Problem:** Some of the algorithms for manipulating RVAs (in particular the complementation procedure) are not usable in practice.

**Solution:** We will show that

- The sets definable in $\langle \mathbf{R}, \mathbf{Z}, +, \leq \rangle$ satisfy some topological properties;
- automata representing such sets have a special structure;
- This special structure makes the use of much simpler algorithms possible.

## Properties of Arithmetic Sets

- On the reals, Boolean combinations of linear (in)equalities define Boolean combinations of open and closed sets.

- The first-order theory of the reals admits quantifier elimination.

- Thus, only Boolean combinations of open and closed sets can be defined in the first-order theory of the reals.

- This should translate to properties of the automata accepting the encodings of these sets.

- However, we are looking at the first-order theory of the reals *and integers* for which no quantifier elimination result is known. Can we say something of the topology of the sets defined in this theory?

## A little Topological Background

Let $S$ be a set and $d(x, y)$ a *distance* defined on the elements of $S$.

- A *neighborhood* of a point $x \in S$ is a set $N_\varepsilon(x) = \{y \in S \mid d(x, y) < \varepsilon\}$, with $\varepsilon > 0$;

- A set $U \subseteq S$ is *open* if for every $x \in U$, there exists $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq U$;

- A set $U \subseteq S$ is *closed* if the set $S \setminus U$ is open;

- The *Borel hierarchy* defines a collection of classes of sets, that starts with the following.
  - The closed sets: $F$;
  - The open sets: $G$;
  - The countable unions of closed sets: $F_\sigma$;
  - The countable intersections of open sets: $G_\delta$;
  - The countable intersections of sets in $F_\sigma$ : $F_{\sigma\delta}$;
  - ...

# The Borel Hierarchy: A Graphical Representation

$$
\begin{tikzcd}
 & \vdots & \\
F_{\sigma\delta} & & G_{\delta\sigma} \\
 & F_{\sigma\delta} \cap G_{\delta\sigma} \arrow[uul] \arrow[uur] & \\
 & \mathcal{B}(F_\sigma) = \mathcal{B}(G_\delta) \arrow[u] & \\
F_\sigma \arrow[ur] & & G_\delta \arrow[ul] \\
 & F_\sigma \cap G_\delta \arrow[ul] \arrow[ur] & \\
 & \mathcal{B}(F) = \mathcal{B}(G) \arrow[u] & \\
F \arrow[ur] & & G \arrow[ul] \\
 & F \cap G \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

- $X \longrightarrow Y \ : \ X \subset Y$;
- $\mathcal{B}(X)$ : Boolean combinations of sets in $X$.

## Topological Properties of Arithmetical Sets

We consider the topology induced by the Euclidean distance

$$d(\vec{x}, \vec{y}) = \left( \sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2}$$

on the vectors of $\mathbf{R}^n$.

**Theorem:** The sets definable in the first-order theory $\langle \mathbf{R}, \mathbf{Z}, +, \leq \rangle$ are in the topological class $F_\sigma \cap G_\delta$.

**Proof:** If $\varphi$ is a formula of $\langle \mathbf{R}, \mathbf{Z}, +, \leq \rangle$ then so is $\neg\varphi$. It is thus sufficient to prove that every definable set is in $F_\sigma$.

Let $\varphi$ be a formula of $\langle \mathbf{R}, \mathbf{Z}, +, \leq \rangle$.

1. Let us replace each variable $x$ appearing in $\varphi$ by $x_I + x_F$, with

   - $x_I$ the integer part of $x$;
   - $x_F$ the fractional part of $x$.

**Example :**

$$(\exists x \in \mathbf{R})\phi \longrightarrow (\exists x_I \in \mathbf{Z})(\exists x_F \in \mathbf{R}) \\ (0 \leq x_F < 1 \wedge \phi[x/x_I + x_F])$$

2. Integer and fractional variables are then separated in the atomic formulas.

   **Example :**

$$\begin{array}{c}(x_I + x_F) = (y_I + y_F) + (z_I + z_F) \;\longrightarrow \\ (x_I = y_I + z_I \wedge x_F = y_F + z_F) \\ \vee \;\; (x_I = y_I + z_I + 1 \wedge x_F = y_F + z_F - 1)\end{array}$$

3. The quantifiers are then distributed over the Boolean operators and unnecessary ones are eliminated.

   **Example :**

$$(Qx_I \in \mathbf{Z})(\phi_I \, \alpha \, \phi_F) \;\longrightarrow\; (Qx_I \in \mathbf{Z})(\phi_I) \;\alpha\; \phi_F,$$

   where

   - $Q \in \{\exists, \forall\}$, $\alpha \in \{\wedge, \vee\}$,
   - $\phi_I$ only contains integer variables,
   - $\phi_F$ only contains fractional variables.

4. One then obtains a formula $\varphi$ of the form

$$\mathcal{B}(\phi_I^{(1)}, \phi_I^{(2)}, \ldots, \phi_I^{(m)}, \phi_F^{(1)}, \phi_F^{(2)}, \ldots, \phi_F^{(m')}).$$

For each value $(a_1, a_2, \ldots, a_k) \in \mathbf{Z}^k$ of the free integer variable of this formula, each subformula $\phi_I^{(i)}$ is identically true or false. One thus has

$$\varphi \equiv \bigvee_{\vec{a} \in \mathbf{Z}^k} \Big( (x_I^{(1)}, \ldots, x_I^{(k)}) = (a_1, \ldots, a_k) \wedge \mathcal{B}_{(a_1,\ldots,a_k)}(\phi_F^{(1)}, \ldots, \phi_F^{(m')}) \Big).$$

The formula $\varphi$ hence defines a countable union of Boolean combinations of open and closed sets, thus a set in $F_\sigma$.

## Automata and the Topology on Words

Consider the topology on infinite words induced by the distance

$$d(w, w') = \frac{1}{|\mathit{commonprefix}(w, w')| + 1}.$$

**Theorem [SW74,MS97] :** The $\omega$-regular languages in the class $F_\sigma \cap G_\delta$ are exactly those accepted by *weak* deterministic automata.

A weak automaton is a Büchi automaton whose set of states can be partitioned into sets $Q_1, Q_2, \ldots, Q_m$ such that

- There exists a partial order $\leq$ among these sets with the property that

$$(\forall q \in Q_i, q' \in Q_j)(q \rightarrow^* q' \;\Rightarrow\; Q_j \leq Q_i);$$

- Each $Q_i$ contains only accepting or nonaccepting states.

The previous result does not guarantee that any automaton built for a set in $F_\sigma \cap G_\delta$ is weak, but we have the following.

**Definition:** An automaton is *inherently weak* is none of its strongly connected components contains both accepting and nonaccepting cycles.

**Theorem:** Any deterministic Büchi automaton accepting an language in $F_\sigma \cap G_\delta$ is inherently weak.

**Proof:**

- For any language $L$ accepted by a deterministic automaton that is not inherently weak, $(\exists w_1)(\forall \varepsilon_1 > 0)(\exists w_2)(\forall \varepsilon_2 > 0)(\exists w_3)\cdots$
  - $d(w_i, w_{i+1}) < \varepsilon_1$ for $i = 1, 2, 3, \ldots$,
  - $w_1, w_3, w_5, \ldots \in L$, and
  - $w_2, w_4, w_6, \ldots \notin L$.
- No language with this property can be accepted by a weak automaton.

## Topology: from Vectors to Words

The topologies on vectors and words are different. To use the fact that we are dealing with sets in $F_\sigma \cap G_\delta$ in the automaton context, we need the following.

**Theorem:** If $S \subseteq \mathbf{R}^n$ is a set in $F_\sigma \cap G_\delta$ (wrt Euclidean distance), then $L_r(S)$ is a set in $F_\sigma \cap G_\delta$ (wrt distance on words).

- The proof has to take into account the fact that every word is not necessarily an encoding of a vector.

- Dual encodings also prevent a direct mapping between the topologies.

- Nevertheless, the proof goes through for the class $F_\sigma \cap G_\delta$.

## Computing with RVAs

From the results we have just seen, it follows that:

**Theorem:** Any deterministic RVA representing a set defined by a formula of the theory $\langle \mathbf{R}, \mathbf{Z}, +, \leq \rangle$ is inherently weak.

This property allows us to work with RVAs that are weak automata and makes possible to use algorithms that are specific to this class of automata.

- *Linear equations and inequations* : The algorithms proposed in [BRW98] produce weak automata.
- *Intersection, union, Cartesian product, projection* : One uses the corresponding operations on languages. The weak nature of the automata is preserved.

- *Complementation* :
  1. The weak RVA is viewed as a *co-Büchi* automaton (a word is accepted if there is an execution of the automaton on that word that does not go infinitely often through accepting states).
  2. For co-Büchi automata, there is a simple determinization procedure (see next slide).
  3. The resulting deterministic automaton is complemented into a Büchi automaton.
  4. The resulting automaton must be inherently weak and hence can easily be transformed into a weak automaton.
- *Satisfiability* : One checks whether the RVA has a reachable accepting strongly connected component.

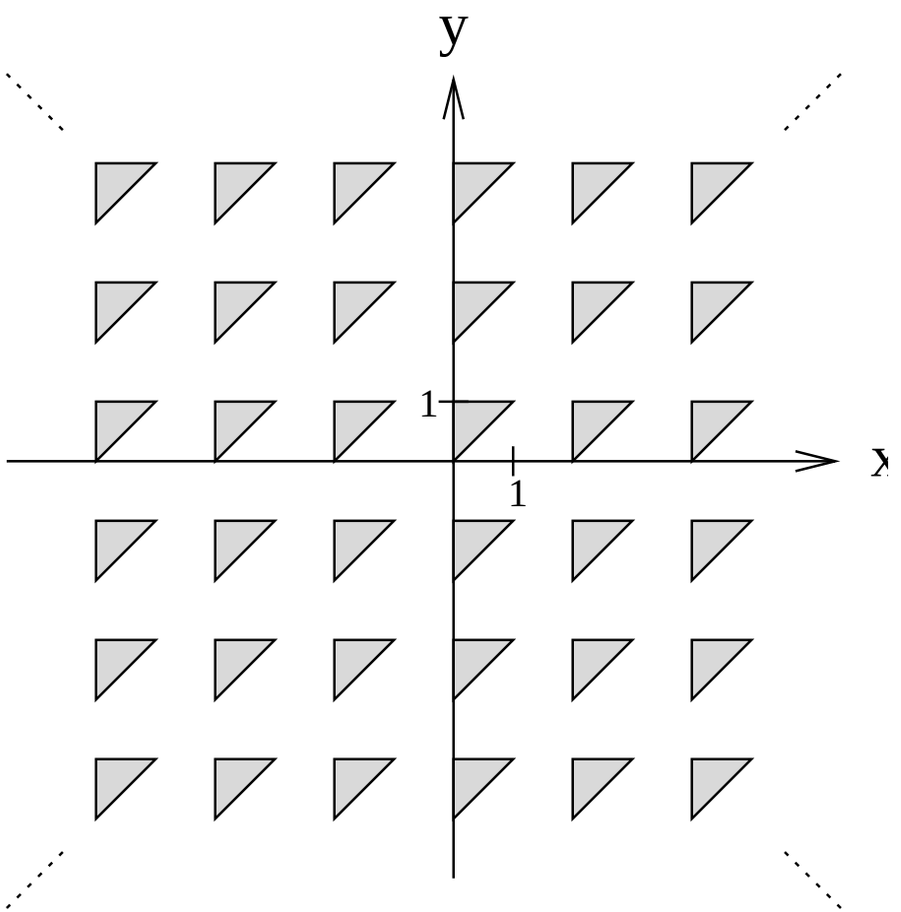
## Determinizing co-Büchi automata

Let $A = (Q, \Sigma, \delta, q_0, F)$ be a nondeterministic co-Büchi automaton. The deterministic co-Büchi automaton $A' = (Q', \Sigma, \delta', q_0', F')$ defined as follows accepts the same $\omega$-language.

- $Q' = 2^Q \times 2^Q$.
- $q_0' = (\{q_0\}, \emptyset)$.
- For $(S, R) \in Q'$ and $a \in \Sigma$, $\delta'$ is defined by
  - if $R = \emptyset$, then $\delta((S, R), a) = (T, T \setminus F)$ where $T = \{q \mid \exists p \in S \text{ and } q \in \delta(p, a)\}$;
  - if $R \neq \emptyset$, then $\delta((S, R), a) = (T, U \setminus F)$ where $T = \{q \mid \exists p \in S \text{ and } q \in \delta(p, a)\}$, and $U = \{q \mid \exists p \in R \text{ and } q \in \delta(p, a)\}$.
- $F' = 2^Q \times \emptyset$.

## An Example

$$\left\{ (x_1, x_2) \in \mathbf{R}^2 \mid \begin{array}{c} (\exists x_3, x_4 \in \mathbf{R}) \\ (\exists x_5, x_6 \in \mathbf{Z}) \end{array} \left( \begin{array}{c} (x_1 = x_3 + 2 \cdot x_5) \wedge \\ (x_2 = x_4 + 2 \cdot x_6) \wedge \\ (x_3 \geq 0 \wedge x4 \leq 1 \wedge x_4 \geq x_3) \end{array} \right) \right\}$$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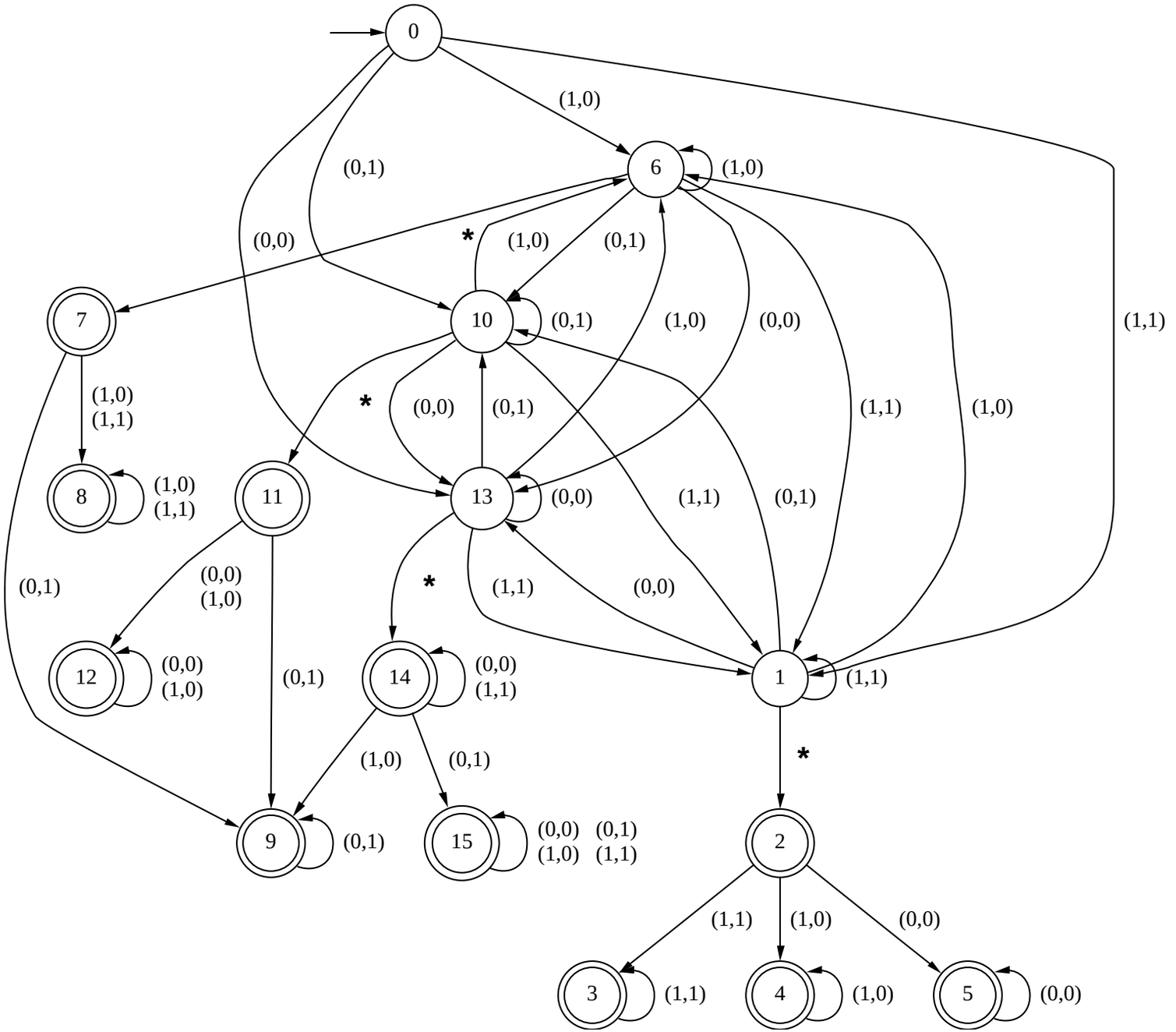
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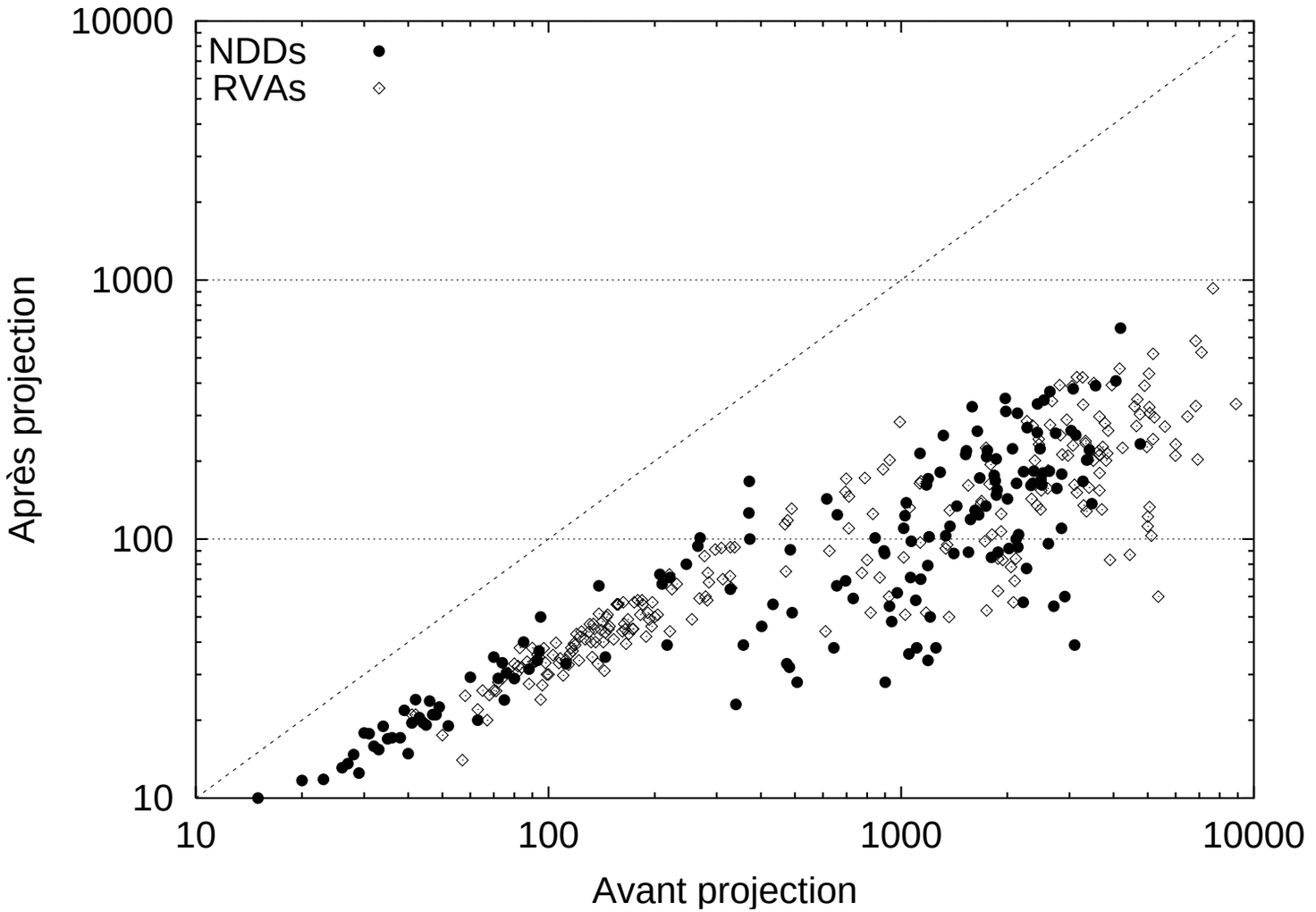
# Performance: the impact of projection and determinization



## Conclusions

- These results do not introduce new algorithms, but show that known algorithms can be used in situations where this was *a priori* impossible.

- Weak deterministic automata have a canonical minimized form [Löding01]. There is thus a canonical form for RVAs.

- From a practical point of view, RVAs seem just as usable as automata representations of sets of integers.

- Experiments with an implementation confirm this.