Geometric Registration for Deformable Shapes

3.2 Isometric Matching and Quadratic Assignment

Quadratic Assignment · Spectral Matching · MRF Model
Overview and Motivation
Global Isometric Matching

Goal

- We want to compute correspondences between deformable shape
- *Global algorithm*, no initialization
Global Isometric Matching

Approach & Problems

- Consistency criterion: global isometry

Problem

- How to find globally consistent matches?

Model

- Quadratic assignment problem
  - General QA-problem is NP-hard
  - But it turns out: solution can usually be computed in polynomial time (more later)
Isometric Matching

(vs. extrinsic matching)
Invariants

Rigid Matching

- Invariants: All Euclidean distances are preserved
Invariants

Intrinsisc Matching

- Invariants: All geodesic distances are preserved
Invariants

Intrinsisc Matching

• Preservation of geodesic distances ("intrinsic distances")
• Approximation
  ▪ Cloth is almost unstretchable
  ▪ Skin does not stretch a lot
  ▪ Most live objects show approximately isometric surfaces
• Accepted model for deformable shape matching
  ▪ In cases where one subject is presented in different poses
  ▪ Across different subjects: Other assumptions necessary
  ▪ Then: global matching is an open problem
Feature Based Matching

Quadratic Assignment Model
Problem Statement

Deformable Matching

- Two shapes: original, deformed
- How to establish correspondences?
- Looking for global optimum
  - Arbitrary pose

Assumption

- Approximately isometric deformation

[data set: S. König, TU Dresden]
Algorithm

Feature-Matching

• Detect feature points

• Local matching: potential correspondences

• Global filtering: correct subset
Algorithm

Feature-Matching

• Detect feature points
  - Locally unique points
  - Such as: maxima of Gaussian curvature
  - E.g.: Geometric MLS-SIFT Features

• Local matching: potential correspondences

• Global filtering: correct subset
Algorithm

Feature-Matching

- Detect feature points
  - Locally unique points
  - Such as: maxima of Gaussian curvature
  - E.g.: Geometric MLS-SIFT Features
- Local matching: potential correspondences
  - Descriptors
  - E.g. curvature histograms
- Global filtering: correct subset
Algorithm

Feature-Matching

• Detect feature points
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• Local matching: potential correspondences
  - Descriptors
  - E.g. curvature histograms

• Global filtering: correct subset
  - Quadratic assignment
  - Spectral relaxation [Leordeanu et al. 05]
  - RANSAC
Quadratic Assignment

Most difficult part: Global filtering

- Find a consistent subset
- Pairwise consistency:
  - Correspondence pair must preserve intrinsic distance
- Maximize number of pairwise consistent pairs
  - Quadratic assignment (in general: NP-hard)
Quadratic Assignment Model

Quadratic Assignment

- \( n \) potential correspondences
- Each one can be turned on or off
- Label with variables \( x_i \)
- Compatibility score:

\[
P^{(\text{match})}(x_1, \ldots, x_n) = \prod_{i=1}^{n} P_i^{(\text{single})} \prod_{i,j=1}^{n} P_{i,j}^{(\text{compatible})}, x_i \in \{0,1\}
\]
Quadratic Assignment Model

Quadratic Assignment

- Compatibility score:
  - Singeltons:
    Descriptor match

\[ P^{(\text{match})}(x_1, \ldots, x_n) = \prod_{i=1}^{n} P^{(\text{single})}_i \prod_{i,j=1}^{n} P^{(\text{compatible})}_{i,j}, x_i \in \{0,1\} \]
Quadratic Assignment Model

Quadratic Assignment

- Compatibility score:
  - **Singletons:** Descriptor match
  - **Doubles:** Compatibility

\[
P^{(match)}(x_1, \ldots, x_n) = \prod_{i=1}^{n} P^{(single)}_i \prod_{i,j=1}^{n} P^{(compatible)}_{i,j}, x_i \in \{0,1\}
\]
Quadratic Assignment Model

Quadratic Assignment

- Matrix notation:
  \[ P^{(match)}(x_1, ..., x_n) = \prod_{i=1}^{n} P^{(single)}_i \prod_{i,j=1}^{n} P^{(compatible)}_{i,j} \]

  \[ \log P^{(match)}(x_1, ..., x_n) = \sum_{i=1}^{n} \log P^{(single)}_i + \sum_{i,j=1}^{n} \log P^{(compatible)}_{i,j} \]

  \[ = \mathbf{x}s + \mathbf{x}^T \mathbf{D} \mathbf{x} \]

- Quadratic scores are encoded in Matrix \( \mathbf{D} \)
- Linear scores are encoded in Vector \( \mathbf{s} \)
- Task: find optimal binary vector \( \mathbf{x} \)
Spectral Matching
Approximate Quadratic Assignment
Spectral Matching

Simple & Effective Approximation:

- Spectral matching [Leordeanu & Hebert 05]
- Form compatibility matrix:

\[ A = \begin{pmatrix}
    a_{11} & a_{21} & a_{31} \\
    a_{12} & a_{22} & \quad \\
    a_{13} & \quad & \quad
\end{pmatrix} \]

Diagonal:
Descriptor match

Off-Diagonal:
Pairwise compatibility

All entries within [0..1]
= [no match...perfect match]
Spectral Matching

Approximate largest clique:

- Compute eigenvector with largest eigenvalue
- Maximizes Rayleigh quotient:

$$\arg\max_x \frac{x^T Ax}{\|x\|^2}$$

- “Best yield” for bounded norm
  - The more consistent pairs (rows of 1s), the better
  - Approximates largest clique

- Implementation
  - For example: power iteration
Spectral Matching

Postprocessing

• Greedy quantization
  ▪ Select largest remaining entry, set it to 1
  ▪ Set all entries to 0 that are not pairwise consistent with current set
  ▪ Iterate until all entries are quantized

In practice...

• This algorithm turns out to work quite well.
• Very easy to implement
• Limited to (approx.) quadratic assignment model
Spectral Matching Example

Application to Animations

- **Feature points:**
  Geometric MLS-SIFT features [Li et al. 2005]

- **Descriptors:**
  Curvature & color ring histograms

- **Global Filtering:**
  Spectral matching

- **Pairwise animation matching:**
  Low precision passive stereo data

Data courtesy of C. Theobald, MPI Informatik
Markov Random Field Model

Probabilistic Interpretation
**Direct MRF Approach**

### Bayesian interpretation

- **Probability Space**
  - \(\Omega = \{ f : (s_1 \ldots s_n) \rightarrow \{1, \ldots, k\}^n \}\)
  - Exponential size!

- **Markov-Random Field / graphical model**

- **Distribution:**
  \[
P(f) = \frac{1}{Z} \left[ \prod_{i=1}^{n} P^{(D)}(s_i, f(s_i)) \right] \left[ \prod_{(i,j) \in G} P^{(S)}(s_i, s_j, f(s_i), f(s_j)) \right]
  \]

  - match local shape
  - preserve local distance
Direct MRF Approach

Solution

- Posterior distribution is \textit{exponential}
- Instead, we compute marginals: "Average" of all solutions

\[
P(f(s_i) = j) = \sum_{i_1=1}^{k} \cdots \sum_{i_n=1}^{k} P(f = (i_1, \ldots, j, \ldots, i_n))
\]

Postprocessing:

- Extract solutions
- Few solutions in a very large space
Direct MRF Approach

Inference

\[ P(f(s_i) = j) = \sum_{i_1=1}^{k} \ldots \sum_{i_n=1}^{k} P(f = (i_1, \ldots, j, \ldots, i_n)) \]

- Representation is polynomial, but computation is still NP hard
- Heuristic approximation: *Loopy belief propagation*
- Works well in practice
Example Result

Self-matching: Deformable Symmetries