Universität
des
Saarlandes
FR 6.2 Informatik

MPI INFORMATIK

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Exercises for Optimization

10. Assignment Due 8.07.2005

Exercise 1 (3 *points*)
Consider the following integer linear program.

$$\begin{array}{lrcrcl} \text{maximize} & x_1 & + & 2x_2 & & \\ \text{subject to} & -3x_1 & + & 4x_2 & \leq & 4 \\ & 3x_1 & + & 2x_2 & \leq & 11 \\ & 2x_1 & - & x_2 & \leq & 5 \\ & & & x_i & \geq & 0, \text{ integer} \end{array}$$

Using a figure, solve the problem by branch and bound. Use the LP relaxations to compute upper bounds, and branch by adding new inequalities $x_i \leq \lfloor x_i \rfloor$ and $x_i \geq \lceil x_i \rceil$ where x_i is a variable with fractional value x_i^*.

Exercise 2 ($2 + 2$ *points*)
Consider an integer linear program of the form

$$\text{maximize } c^T x \text{ subject to } Ax \leq b, x \in \mathbb{Z}^n \tag{1}$$

where A, b, c have integer entries.
Prove or disprove each of the following statements.

a) If (1) is infeasible then its linear relaxation can be feasible and unbounded.

b) If (1) is feasible and bounded then its linear relaxation can be unbounded.

Hint. You may use the following fact. If the linear relaxation of (1) is unbounded then there are $x, y \in \mathbb{Q}^n$ such that $c^T y > 0$ and $x + \lambda y$ is feasible for all $\lambda \geq 0$.

Exercise 3 (3^* *points*)
Given a collection of finite sets $S_1, \ldots, S_n$, the Set Cover Problem asks for a minimum cardinality cover $C \subseteq [n]$ such that

$$\bigcup_{i \in [n]} S_i = \bigcup_{i \in C} S_i$$

Show how to compute in polynomial time a cover $\widetilde{C} \subseteq [n]$ of cardinality at most $f|C^*|$ where $C^* \subseteq [n]$ is a minimum cardinality cover and

$$f = \max_x |\{i \in [n] \mid x \in S_i\}|$$

is the maximum frequency.

Hint. Formulate the problem as an ILP. Consider an optimal solution to the linear relaxation of this ILP. Now round this fractional solution in a simple manner to a feasible integer solution.

Exercise 4 $(1 + 2 + 2^*$ *points*)
Given a directed graph $D = (V, A)$ and to vertices s and t we want to determine the maximum number of arc-disjoint paths from s to t. Let $\mathcal{P} \subseteq 2^A$ denote the set of paths from s to t in D. Consider the following integer linear program.

$$\text{maximize} \sum_{P \in \mathcal{P}} x_P \text{ subject to } \sum_{P \in \mathcal{P} : a \in P} x_P \leq 1 \text{ for all arcs } a \in A,\ x_P \in \mathbb{Z}_{\geq 0} \tag{2}$$

a) Form the dual of the linear relaxation of (2).

b) Show that the dual can be solved in polynomial time using the ellipsoid method.

c) Suppose you want to solve the linear relaxation of (2) using the revised simplex method. Show that one iteration can be done in polynomial time.

Hint. For b) you may use the following fact. A linear program with a polynomial number of variables can be solved in polynomial time if the separation problem for the program can be solved in polynomial time. Given a linear program and a point $\bar{x}$, the separation problem asks for the following: Either descide that $\bar{x}$ is feasible or find a inequality $a^T x \leq \beta$ which is valid for all feasible points but $a^T \bar{x} > \beta$.

Exercise 5 (3 *points*)

Given a graph $G = (V, E)$ with non-negative edge costs $c : E \to \mathbb{R}_{\geq 0}$ and a vertex subset $S \subseteq V$, the Steiner Tree Problem asks for a tree $T \subseteq E$ of minimum cost which connects all vertices in S.
Formulate the Steiner Tree Problem as an integer linear program.

Exercise 6 $(3 + 4^* + 4$ *points*)

a) Given a sequence $x_1, \ldots, x_n$ of real numbers, show that there is a sequence $y_1, \ldots, y_n$ with $y_i \in \{\lfloor x_i \rfloor, \lceil x_i \rceil\}$ such that the error in each interval is less than 1, i.e., for all intervals $I \subseteq [n]$ we want

$$\left| \sum_{i \in I} x_i - y_i \right| < 1.$$

Find an algorithm that computes such a sequence (y_i) in linear time.

Remark. You get bonus points if you can improve on the bound of 1. An efficient algorithm which achieves this bound is not required.

b) Given a sequence $x_1, \ldots, x_n$ of real numbers, show that there is a rounding $y_1 \ldots, y_n$ of (x_i) such that the error in each interval with respect to two orders is less than 2, i.e., for a fixed permutation π of $[n]$ and all intervals $I \subseteq [n]$ we want

$$\left| \sum_{i \in I} x_i - y_i \right| < 2 \text{ and } \left| \sum_{i \in I} x_{\pi(i)} - y_{\pi(i)} \right| < 2.$$

Hint. First show that the error is less than 1 for each interval I with $1 \in I$.