Universität
des
Saarlandes
FR 6.2 Informatik

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Exercises for Optimization

2. Assignment Due 6.05.2005

Exercise 1 (5 *Points*)

Consider the classic *diet problem*: Various foods are available, each unit of which contributes a certain amount toward the minimum daily requirements of various nutritional needs. Each food has a particular cost. The goal is to choose how many units of each food to purchase to meet the minimum daily nutritional requirements, while minimizing the total cost. Formulate this as a linear program, state the dual problem and give an "economic interpretation" of the dual.

Exercise 2 ($2 + 3$ *Points*)

a) Show that for each linear program of the form

$$\text{maximize } c^T x \text{ subject to } Ax \leq b$$

there is an equivalent linear program of the form

$$\text{maximize } c'^T x' \text{ subject to } A'x' = b', x' \geq 0.$$

b) Show that the following two problems are polynomial-time equivalent.

- Given a linear program, find a feasible solution or decide that there is none.
- Given a linear program, find an optimal solution or decide that there is none.

Exercise 3 ($3 + 2$ *Points*)

a) Derive and prove a Theorem of the Alternatives for the system

$$Ax = b, x \geq 0.$$

b) Let $S = \{a_1, \ldots, a_k\} \subset \mathbb{R}^n$ and let

$$\text{conv}(S) := \left\{ \sum_{i=1}^{k} \lambda_i a_i \mid \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1 \right\} \subset \mathbb{R}^n$$

denote the convex hull of S.

Show that for each $b \in \mathbb{R}^n$ either b is contained in conv(S) or there exists a closed half-space $H \subset \mathbb{R}^n$ with conv$(S) \subseteq H$ and $b \notin H$.

Exercise 4 $(3+2+2^*+2^*$ *Points*$)$

Note that you earn bonus points for 4.c) and 4.d).

a) Prove or disprove that the following sets are (finite) intersections of halfspaces.

   (a) $\{x \in \mathbb{R}^n \mid |x_i| \leq 1\}$
   (b) $\{x \in \mathbb{R}^n \mid \sum_{i=1}^n |x_i| \leq 1\}$
   (c) $\{x \in \mathbb{R}^n \mid x_1 > 0\}$

b) Consider the piecewise linear function $f(x) = \min_j f^{(j)}(x)$ for affine linear functions $f^{(j)}(x) = d^{(j)} + \sum_i c_i^{(j)} x_i$. Reduce the following problem to an equivalent linear programming problem.

$$\text{maximize } f(x) \text{ subject to } Ax \leq b.$$

c) Let $H = \{x \in \mathbb{R}^n \mid a^T x \leq \beta\}$ be a half-space and $P = \{x \in \mathbb{R}^n \mid a^T x = \beta\}$ the corresponding hyperplane. Define $d_P(x) := \min_{y \in P} \|x - y\|_2$.

   Show that $d_P(x)$ is a affine linear function if x ranges over H.

   (*Hint.* Write $x = \frac{a^T x}{a^T a} a + x'$ and $y = \frac{\beta}{a^T a} a + y'$ with $a^T x' = a^T y' = 0$. Then, minimize $\|x - y\|_2^2$.)

d) Suppose $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}$. Let $S := \{x \in \mathbb{R}^n \mid Ax \leq b\}$. We want to find a ball contained in S with maximum radius. Formulate this as a linear program.

   (*Hint.* $\{y \mid \|y - x\|_2 \leq r\}$ is the (closed) ball with center x and radius r.)