I N F O R M A T I K

Universität des Saarlandes
FR 6.2 Informatik

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Exercises for Optimization

3. Assignment Due 13.05.2005

Exercise 1 (3 *Points*)
Given the linear program

$$(P):\ \text{minimize } c^T x \text{ subject to } Ax = b,$$

use the dual program to show that exactly one of the following 3 possibilities holds:

- (P) is infeasible,
- (P) is unbounded,
- all feasible solutions of (P) are optimal.

Exercise 2 (3 *Points*)
Derive and prove a Theorem of the Alternatives for the system

$$Ax < b.$$

(*Hint.* Consider the problem to maximize t subject to $Ax + t\mathbb{1} \leq b$ and its dual problem. $\mathbb{1}$ is the all-ones vector.)

Exercise 3 (3 *Points*)
Is $x^* = (0, 4/3, 2/3, 5/3, 0)^T$ an optimal solution of (P)? (*Hint.* Use complementary slackness.)

$$(P)\quad \begin{array}{rrcrcrcrcrl} \max & 7x_1 & + & 6x_2 & + & 5x_3 & - & 2x_4 & + & 3x_5 & \\ s.t. & x_1 & + & 3x_2 & + & 5x_3 & - & 2x_4 & + & 2x_5 & \leq 4 \\ & 4x_1 & + & 2x_2 & - & 2x_3 & + & x_4 & + & x_5 & \leq 3 \\ & 2x_1 & + & 4x_2 & + & 4x_3 & - & 2x_4 & + & 5x_5 & \leq 5 \\ & 3x_1 & + & x_2 & + & 2x_3 & - & x_4 & - & 2x_5 & \leq 1 \\ & & & & & & & & & x_i & \geq 0 \end{array}$$

Exercise 4 (3 *Points*)
Formulate and prove the theorem of complementary slackness for the following linear program and its dual program:

$$\begin{array}{rrcrl} \max & c^T x & + & d^T y & \\ s.t. & Ax & + & By & \leq a \\ & Cx & + & Dy & = b \\ & x & & & \geq 0 \end{array}$$

You may use the rules for forming the dual and the theorem of strong duality.

Exercise 5 (2 + 2 *Points*)
We are given the following optimization problem:
A factory is producing two types of (waist-)belts A and B. A is of better quality than B. The profit is 2 for type A and 1,50 for type B. The production of one belt of type A needs 2 units of time whereas only 1 time unit is needed to produce one belt of type B. There is a daily capacity of 1000 units of time. The leather delivery allows a production of 800 belts per day. There are at most 400 belt buckles for type A per day available and 700 for type B.
Which amount of each type A and B should be produced to maximize the profit?

a) Formulate this problem as a linear program.

b) Solve this problem graphically.

Exercise 6 ($2 + 2 + 2^* + 2^* + 2^*$ *Points*)
Let $D = (V, A)$ be a directed graph with capacities $c : A \to \mathbb{R}_{\geq 0}$, source $s \in V$ and sink $t \in V$.
Suppose each arc $a \in A$ can carry a non-negative amount of flow of at most $c(a)$. The total amount of flow entering a vertex $v \neq s, t$ should be equal to the total amount of flow leaving v.
The *maximum flow problem* asks for maximizing the net amount of flow leaving s, that is the difference between the amount of flow leaving s and the amount of flow entering s.
The following notation may be useful.

$$\begin{aligned} \delta^-(U) &:= \text{ set of arcs entering } U \subseteq V \text{ in } D \\ \delta^+(U) &:= \text{ set of arcs leaving } U \subseteq V \text{ in } D \\ \delta^{\pm}(u) &:= \delta^{\pm}(\{u\}) \text{ for } u \in V \end{aligned}$$

You earn bonus points for c)–e).

a) Formulate the maximum flow problem as a linear program (P).

b) Determine the corresponding dual program (D).

c) Let x^* and y^* be optimal solutions for (P) and (D), respectively.
Formulate the complementary slackness conditions for x^* and y^*.

d) Show that there exists W with $s \in W \subseteq V - t$ such that all arcs $\delta^-(W)$ carry no flow and each arc $a \in \delta^+(W)$ carries flow $c(a)$ in x^*. (*Hint.* Use complementary slackness.)

e) Let OPT denote the objective function value of x^*. Now show

$$OPT = \sum_{a \in \delta^+(W)} c(a) \leq \sum_{a \in \delta^+(U)} c(a) \text{ for all } U \text{ with } s \in U \subseteq V - t.$$

Remark. This fact is called 'max-flow min-cut' theorem.