INFORMATIK

Universität des Saarlandes
FR 6.2 Informatik

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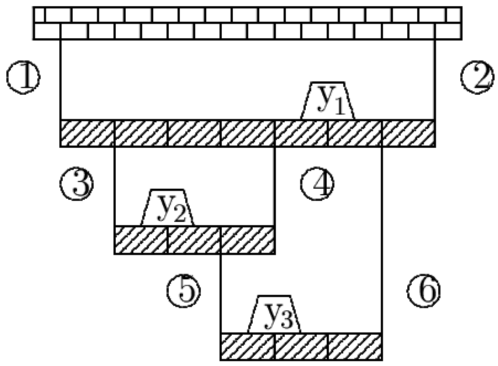
Exercises for Optimization

4. Assignment Due 20.05.2005

Exercise 1 (2 *Points*)
Given a linear program with n free variables, can you find an equivalent program with $n+1$ nonnegative variables?

Exercise 2 (3 *Points*)
Consider the following scaffold.



Cable 1 and 2 can bear a loading of 300 kg each, cable 3 and 4 only 100 kg each, and cable 5 and 6 only 50 kg each. Neglecting the weight of cables and planks, we are searching for the maximal allowed total weight $y_1 + y_2 + y_3$. Formulate this problem as a linear program.

Exercise 3 (3 *Points*)
Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ for some $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$ and let $x^* \in P$.
We are interested in the objective functions $P \ni x \mapsto c^T x$ that attain their maximum at x^*.
Show that $C := \{c \in R^n \mid c^T x^* = \max_{x\in P} c^T x\}$ is a cone generated by rows of A, i.e.,

$$C = \left\{ \sum_{i\in I} y_i \, a_i \mid y_i \geq 0 \right\}$$

for some index set $I \subseteq \{1, \ldots, m\}$ where a_i^T is the i-th row of A.

Remark. In general, the cone generated by the vectors $x_1, \ldots, x_m$ is the set

$$\text{cone}\{x_1, \ldots, x_m\} := \left\{ \sum \lambda_i x_i \mid \lambda_1, \ldots, \lambda_m \geq 0 \right\}.$$

Exercise 4 $(2+1+2+2$ *Points*$)$
Consider the following linear program.

$$\begin{array}{lrcrcrcrcrl}
\max & -2x_1 & + & 1.5x_2 & + & x_3 & & & & & \\
 & 2x_1 & + & 2x_2 & + & 6x_3 & + & x_4 & & & = 18 \\
 & -2x_1 & + & 3x_2 & + & 4x_3 & + & & & x_5 & = 12 \\
 & & & & & & & & & x_i & \geq 0
\end{array}$$

a) Identify all basic solutions and determine the value of the objective function for all basic feasible solutions. What is the optimal solution.

b) Which pairs of feasible basic solutions are adjacent.

c) Sketch the polyhedron

$$P = \left\{ x \in \mathbb{R}^3_{\geq 0} \mid \begin{pmatrix} 2 & 2 & 6 \\ -2 & 3 & 4 \end{pmatrix} x \leq \begin{pmatrix} 18 \\ 12 \end{pmatrix} \right\}.$$

To each basic feasible solutions of the linear program give the corresponding vertex of the polyhedron.

d) Let x^* be the vertex of P corresponding to the optimal solution of the linear program. Originating at x^*, draw the vectors which generate the cone of all vectors c such that $c^T x^* = \max_{x \in P} c^T x$. (cf. Exercise 3)

Exercise 5 $(3+3^*+2$ *Points*$)$
Let $\emptyset \neq X \subseteq \mathbb{R}^d$ be of finite cardinality and let $u, v, w \in \operatorname{conv}(X)$ arbitrary.

a) Prove Caratheodory's Theorem which is

$$\exists x_0, \ldots, x_d \in X.\ u \in \operatorname{conv}\{x_0, \ldots, x_d\}.$$

b) Also prove the following extension of Caratheodory's Theorem. *(Bonus)*

$$\exists x_1, \ldots, x_d \in X.\ u \in \operatorname{conv}\{v, x_1, \ldots, x_d\}.$$

c) Is the following statement true in general?

$$\exists x_2, \ldots, x_d \in X.\ u \in \operatorname{conv}\{v, w, x_2, \ldots, x_d\}.$$

(*Hint.* For a) and b) try to find corresponding statements about polyhedra of the form

$$\{x \mid Ax = b, \sum x_i = 1, x \geq 0\}.$$

If you think c) is not true in general then try to find a *simple* counterexample.)