I N F O R M A T I K

Universität des Saarlandes
FR 6.2 Informatik

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Exercises for Optimization

5. Assignment — Due 3.06.2005

Exercise 3 (4 *Points*)
Solve the following linear program using the simplex method:

$$
\begin{array}{lrcrcrcrcrcr}
\min & -5x_1 & + & -4x_2 & & & & & & & & \\
 & 2x_1 & + & 3x_2 & + & x_3 & & & & & = & 12 \\
 & 4x_1 & + & x_2 & + & & & x_4 & & & = & 16 \\
 & x_1 & + & x_2 & + & & & & & x_5 & = & 4.3 \\
 & & & & & & & & & x_i & \geq & 0
\end{array}
$$

Exercise 2 ($2+2+1$ *Points*)
A physicist takes measurements of a variable $y(x)$; the results are in the form of pairs (x_i, y_i). The physicist wishes to find the straight line that fits this data best in the sense that the maximum vertical distance between any point (x_i, y_i) and the line is as small as possible

a) Formulate this problem as a linear program.

b) Form the dual linear program.

c) Suppose you want to solve the problem by the simplex method. Would you prefer to solve the primal or the dual program?

Exercise 3 ($2+2+2+2^*$ *Points*)
Let (P) be a bounded and feasible linear program in standard form with linearly independent constraints and let (D) be its dual problem. Prove or disprove each of the following statements.

a) If all basic feasible solutions of (P) are nondegenerate then (P) has a unique optimal solution.

b) If all optimal basic solutions of (P) are nondegenerate then (D) has a unique optimal solution.

c) If one optimal basic solution of (D) is nondegenerate then (P) has a unique optimal solution.

d) If (P) has a unique optimal solution then all optimal basic solutions of (D) are nondegenerate.

Exercise 4 $(1+2+2+1+3+1$ *Points*$)$
Let $S^k := \{w \in \mathbb{R}^k \mid \sum w_i = 1, w \geq 0\}$ denote the convex hull of the unit vectors in $\mathbb{R}^k$ for all $k \in \mathbb{N}$ and let A be an $n \times m$-matrix.

a) Show that

$$f : S^m \to \mathbb{R}, \quad u \mapsto \min_{v \in S^n} v^T A u$$

is a piecewise linear function, i.e., there are linear functions $f^{(1)}, \dots, f^{(n)}$ such that

$$f(u) = \min_{1 \leq j \leq n} f^{(j)}(u).$$

b) Certain two-person games can be formulated mathematically as the problem to solve

$$\max_{u \in S^m} \min_{v \in S^n} v^T A u.$$

Formulate this problem as a linear program.

c) Form the dual of this linear program.

d) Show von Neumann's Minimax Theorem which is

$$\max_{u \in S^m} \min_{v \in S^n} v^T A u = \min_{v \in S^n} \max_{u \in S^m} v^T A u.$$

e) Consider the following game. Roberta (R) and Charles (C) put their hands behind their backs and make a sign for one of the following: stone (1), paper (2) and scissors (3). They then simultaneously display their chosen sign. The winner is determined by the following rules: paper beats stone, scissors beats paper and stone beats scissors. The outcome is a draw when the two players choose the same sign. Player R pays a certain amount of money to C according to the following matrix A. Negative amounts indicate that R actually receives money from C.

	A:	C: 1 (Scissors)	2 (Stone)	3 (Paper)
R:	1 (Scissors)	0	1	−3
	2 (Stone)	−2	0	2
	3 (Paper)	3	−1	0

Suppose C chooses sign j with probability u_j and R chooses sign i with probability v_i. Since they display their chosen signs simultaneously, the events of choosing signs are independent. The expected payoff for C is

$$\mathbb{E}[\text{payoff}] = \sum_{i=1}^{3} \sum_{j=1}^{3} v_i A_{ij} u_j = v^T A u$$

We call u and v (mixed) strategies of R and C, respectively.

- Explain why a maximizer of $S^3 \ni u \mapsto \min_{v \in S^3} v^T A u$ might be a good strategy for C.
- Show that $\max_{u \in S^3} \min_{v \in S^3} v^T A u = 0$. (*Hint.* The simplex method might not be the easiest way.)

f) Give an interpretation of d) with respect to e).