INFORMATIK

Universität des Saarlandes
FR 6.2 Informatik

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Exercises for Optimization

6. Assignment Due 10.06.2005

Exercise 1 (2 *Points*)
Consider the following linear programs.

$$(P^+): \text{ minimize } \quad c^T x \text{ subject to } Ax = b, x \geq 0$$

$$(P^-): \text{ minimize } -c^T x \text{ subject to } Ax = b, x \geq 0$$

Is it possible that both (P^+) and (P^-) are unbounded?

Exercise 2 $(1 + 2 + 1 + 1$ *Points*$)$
For $A \in \mathbb{R}^{m \times n}, c \in \mathbb{R}^n, b \in \mathbb{R}^m$ consider the following linear program

$$(LP): \text{ maximize } c^T x \text{ subject to } Ax \leq b, x \geq 0$$

a) Show that (LP) is feasible if $b \geq 0$.

b) Transform (LP) to a linear program (LP') in standard form

$$(LP'): \text{ minimize } c'^T x' \text{ subject to } A'x' = b, x' \geq 0$$

by introducing m slack variables and show that (LP') has a feasible basis B^0 if $b \geq 0$. Determine the simplex tableau for this basis.

c) Consider the simplex tableau for an arbitrary feasible basis B of (LP'). Show that the columns of the slack variables form the inverse of the basis matrix ${A'_B}^{-1}$.

d) Find a mapping from the basic feasible solutions of (LP) to the basic feasible solutions of (LP'). Show that two basic feasible solutions of (LP) are adjacent if and only if the corresponding basic feasible solutions of (LP') are adjacent.

Exercise 3 $(3 + 2 \times 2 + 3 \times 2^*$ *Points*$)$
A furniture industry produces shelfs, desks, chairs, and beds. In order to produce one product unit $3, 2, 1$ and 2 working hours are necessary, respectively. Furthermore $4, 3, 3$ and 4 units of timber are needed, respectively. One unit of metal is needed in each case. 225 working hours, 117 units of metal, and 420 units of timber are available. The producer yields a profit of 19 per shelf, 13 per desk, 12 per chair and 17 per bed.

a) Determine an optimal production plan using the simplex tableau.

b) Use the methods of sensitivity analysis to determine an optimal production plan if one of the following changes has to be considered additionally.

(a) The profit per desk increases from 13 to 15.

(b) The available stock of metal increases from 117 units to 125 units.

(c) There is an additional production of tables that yield a profit of 14 each. In order to produce one table, 3 working hours, 1 unit of metal, and 2 units of timber are needed.

(d) The number of produced chairs should be at most five times the number of produced desks.

(e) To produce one chair, 4 instead of 3 units of timber are needed.

Exercise 4 ($2 + 2 + 2 + 5^*$ *Points*)
Let $P = \{x \mid Ax \leq b, x \geq 0\} \subseteq \mathbb{R}^n$ be a bounded non-empty polyhedron and let $G(P) = (V(P), E(P))$ be a graph where $V(P)$ are the vertices of P and $E(P)$ consists of all edges uv such that u and v are two adjacent vertices of P.

a) Show that $G(P)$ is connected. Given vertices u and v, how can a path from u to v in $G(P)$ be computed?

b) Suppose $\min_{x \in P} c^T x$ has the unique optimal solution x^*. Show that the vertices with second best objective function value are neighbors of x^* in $G(P)$.

c) Suppose $a^T x \leq \beta$ is a valid inequality for P, i.e.,

$$\forall x \in P.\ a^T x \leq \beta.$$

For $H = \{x \in \mathbb{R}^n \mid a^T x = \beta\}$ show that $G(P \cap H)$ is the subgraph of $G(P)$ induced by the vertices $V(P) \cap H$, i.e., show

- $P = \{x \mid Ax \leq b, x \geq 0, a^T x \leq \beta\}$,
- a vertex of $P \cap H$ is a vertex of P,
- two vertices u, v of $P \cap H$ are adjacent if and only if they are adjacent in P.

d) Show that $G(P)$ is n-(vertex-)connected if P is full-dimensional, i.e., if P is not contained in any hyperplane.

Remark.
$G(P)$ is n-connected if $G(P) \backslash W$ is connected for each vertex set $W \subseteq V(P)$ with $|W| = n - 1$.

Hints.

- For $u, v, w_1, \ldots, w_{n-1} \in V(P)$ show the following: If there is an obj. function c such that

$$\max_{1 \leq i \leq n-1} c^T w_i \leq \min\{c^T u, c^T v\} \tag{1}$$

$$\max_{1 \leq i \leq n-1} c^T w_i < \max_{x \in P} c^T x \tag{2}$$

then there is a path from u to v in $G(P)$ that does not use any vertex w_i, i.e., u and v are connected in $G(P) \backslash W$.

- Now show that for any $u, v, w_1, \ldots, w_{n-1} \in V(P)$ there exists $c \in \mathbb{R}^n$ such that

$$c^T w_i = c^T u \text{ for all } 1 \leq i < n$$
$$c^T w_i \leq c^T v \text{ for all } 1 \leq i < n$$
$$c^T w_i < \max_{x \in P} c^T x \text{ for all } 1 \leq i < n$$

which clearly implies (1) and (2).