INFORMATIK

Universität des Saarlandes
FR 6.2 Informatik

Dr. Ernst Althaus, Dr. Benjamin Doerr, David Steurer SS 2005

Exercises for Optimization

7. Assignment Due 17.06.2005

Exercise 1 (3 *Points*)
Show that a non-empty polyhedron of the form

$$\{x \mid Ax = b, x \geq 0\}$$

is unbounded if and only if

$$\{y \mid Ay = 0, y \geq 0\} \neq \{0\}.$$

Hint. Use the simplex method.

Exercise 2 $(1 + 3 * + 3^*$ *Points*$)$

a) Show

$$\left|\frac{p}{q} - \frac{p'}{q'}\right| = 0 \text{ or } \left|\frac{p}{q} - \frac{p'}{q'}\right| > M^{-2}$$

for integers p, p', q, q' with $0 < q, q' < M$.

b) Suppose you are given an algorithm $\mathcal{A}$ which computes a basic feasible solution of a given linear program with n variables and m constraints in time $\mathcal{O}(f(n+m))$. Assume $\mathcal{O}(f(n+m+1)) = \mathcal{O}(f(n+m))$.

Invent an algorithm $\mathcal{B}$ which computes an optimal basic solution of a given linear program

$$\text{minimize } c^T x \text{ subject to } Ax = b, x \geq 0$$

in time $\mathcal{O}(f(n+m)\log M)$ where the entries of A, b, c are integers and M fullfills the following conditions

- $M \geq \sum_{i=1}^{n} |c_i|$,
- $M \geq \|x\|_\infty$ for all basic feasible solution x and
- for any basic feasible solution x of the given linear program there is an $L \leq M$ such that Lx is an integer vector.

Hint. Use binary search.

c) Does your algorithm $\mathcal{B}$ still work if $\mathcal{A}$ computes feasible solutions which are not necessarily basic? Show that an basic feasible solution can be computed in polynomial time, given any feasible solution.

Hint. Recall the proof of the fact that a non-empty polyhedron without a line contains a basic feasible solution. You can assume that Gaussian Elimination runs in polynomial time.

Exercise 3 $(2 + 2 + 2$ *Points*$)$

a) For a graph $G = (V, E)$, a vertex set $S \subseteq V$ is called *vertex cover* if all edges of G are incident to at least one vertex of S. Formulate the problem of finding a minimum cardinality vertex cover in G as an integer linear program.

b) For a graph $G = (V, E)$, a edge set $M \subseteq E$ is called *matching* if no two edges of M are adjacent, i.e., $e \cap f = \emptyset$ for all distinct edges $e, f \in M$. Formulate the problem of finding a maximum cardinality matching in G as an integer linear program.

c) What is the relation between the linear relaxations of the two integer linear programs? Explain why your observation implies that the maximum cardinality of a matching is at most the minimum cardinality of a vertex cover in G.

Exercise 4 $(4 + 3$ *Points*$)$
Given a graph $G = (V, E)$, a function $\chi : V \to \mathbb{N}_{>0}$ is called *coloring* if $\chi^{-1}(i) := \{v \in V \mid \chi(v) = i\}$ is an independent set in G for all $i \in \mathbb{N}$.

a) Formulate the problem to find a coloring χ such that $\max_{v \in V} \chi(v)$ is minimum as an integer linear program.

b) The smallest number k such that there exists a coloring χ with $\chi(v) \leq k$ is called *chromatic number* of G and is denoted by $\chi(G)$.

   Show $\chi(G) \leq \alpha(\overline{G})$ where $\overline{G} = (V, \overline{E})$ and $e \in \overline{E} \Leftrightarrow e \notin E$.