Exercise 1 (2 *Points*)
Given a linear program with n free variables, can you find an equivalent program with $n+1$ nonnegative variables?
Solution:
Suppose we are given the following LP

$$(LP) : \max c^T x \text{ subject to } x \in P = \{x \mid Ax \leq b\} \subseteq \mathbb{R}^n.$$

First assume that P is bounded. Then we can move the polyhedron in the right direction until it is fully contained in $\{x \mid x \geq 0\}$, i.e., there is a real number $M \geq 0$ such that

$$P + M\mathbb{1} = \{x + M\mathbb{1} \mid x \in P\} = \{y \mid y - M\mathbb{1} \in P\} \subseteq \mathbb{R}^n_{\geq 0}.$$

In general, y and M should be the new variables and we get

$$(LP') : \max c^T(y - M\mathbb{1}) \text{ subject to } A(y - M\mathbb{1}) \leq b.$$

Formally,

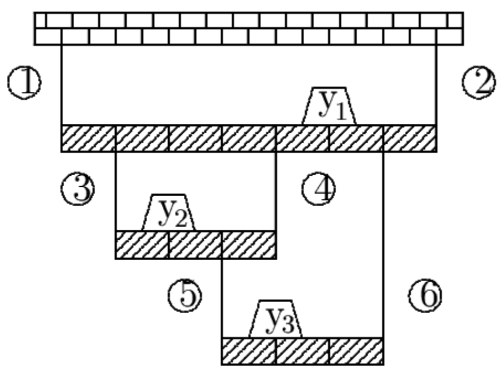
$$x \mapsto (y(x), M(x)) \text{ with } y(x) = x + \|x\|_\infty \mathbb{1} \geq 0 \text{ and } M(x) = \|x\|_\infty \geq 0$$

maps a feasible solution of (LP) to a feasible solution of (LP') with the same objective function value and

$$(y, M) \mapsto x = y - M\mathbb{1}$$

maps a feasible solution of (LP') to a feasible solution of (LP) with the same objective function value.

Exercise 2 (3 *Points*)
Consider the following scaffold.



Cable 1 and 2 can bear a loading of 300 kg each, cable 3 and 4 only 100 kg each, and cable 5 and 6 only 50 kg each. Neglecting the weight of cables and planks, we are searching for the maximal allowed total weight $y_1 + y_2 + y_3$. Formulate this problem as a linear program.
Solution:
We introduce variables $z_1, \ldots, z_6$ for each cable. First we get the constraint

$$z_1, z_2 \leq 300$$
$$z_3, z_4 \leq 100$$
$$z_5, z_6 \leq 50$$

Since the sum of the (directed) forces at each plank should be zero, we get the following constraints.

$$
\begin{aligned}
z_1 + z_2 &= z_3 + z_4 + y_1 + z_6 \\
z_3 + z_4 &= y_2 + z_5 \\
z_5 + z_6 &= y_3
\end{aligned}
$$

Due to the distribution of the weights and cables at the planks, by the principle of levers ("Hebelgesetz") we get

$$
\begin{aligned}
z_1 &= \frac{6z_3 + 3z_4 + 2y_1 + z_6}{7} \\
z_3 &= \frac{2y_2 + z_5}{3} \\
z_5 &= \frac{2y_3}{3}
\end{aligned}
$$

The objective is of course to

$$\text{maximize } y_1 + y_2 + y_3$$

Exercise 3 (3 *Points*)
Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ for some $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$ and let $x^* \in P$.
We are interested in the objective functions $P \ni x \mapsto c^T x$ that attain their maximum at x^*.
Show that $C := \{c \in R^n \mid c^T x^* = \max_{x \in P} c^T x\}$ is a cone generated by rows of A, i.e.,

$$C = \left\{ \sum_{i \in I} y_i \, a_i \mid y_i \geq 0 \right\}$$

for some index set $I \subseteq \{1, \ldots, m\}$ where a_i^T is the i-th row of A.

Remark. In general, the cone generated by the vectors $x_1, \ldots, x_m$ is the set

$$\text{cone}\{x_1, \ldots, x_m\} := \left\{ \sum \lambda_i x_i \mid \lambda_1, \ldots, \lambda_m \geq 0 \right\}.$$

Solution:
Let c be a objective function vector. Then the dual linear program is

$$\min y^T b \text{ subject to } y^T A = c^T, y \geq 0$$

By complementary slackness $x^* \in P$ is optimal if and only if the following system (I) is feasible

$$
\begin{aligned}
y^T A &= c^T \\
y &\geq 0 \\
0 &= y^T (b - Ax^*)
\end{aligned}
$$

where the first two constraints ensure that y is a feasible solution of the dual and the third constraint is equivalent to $y_i = 0$ for all i such that $(b - Ax)_i \neq 0$. Alltogether, system (I) is feasible if and only if

$$c \in C = \left\{ \sum_{i \in I} y_i \, a_i \mid y_i \geq 0 \right\} \text{ for } I = \{i \mid (b - Ax)_i = 0\}$$

Exercise 4 $(2+1+2+2$ *Points*$)$
Consider the following linear program.

$$\begin{array}{lcccccccccc} \max & -2x_1 & + & 1.5x_2 & + & x_3 & & & & & \\ & 2x_1 & + & 2x_2 & + & 6x_3 & + & x_4 & & & = 18 \\ & -2x_1 & + & 3x_2 & + & 4x_3 & + & & & x_5 & = 12 \\ & & & & & & & & & x_i & \geq 0 \end{array}$$

a) Identify all basic solutions and determine the value of the objective function for all basic feasible solutions. What is the optimal solution.

b) Which pairs of feasible basic solutions are adjacent.

c) Sketch the polyhedron

$$P = \left\{ x \in \mathbb{R}^3_{\geq 0} \mid \begin{pmatrix} 2 & 2 & 6 \\ -2 & 3 & 4 \end{pmatrix} x \leq \begin{pmatrix} 18 \\ 12 \end{pmatrix} \right\}.$$

To each basic feasible solutions of the linear program give the corresponding vertex of the polyhedron.

d) Let x^* be the vertex of P corresponding to the optimal solution of the linear program. Originating at x^*, draw the vectors which generate the cone of all vectors c such that $c^T x^* = \max_{x \in P} c^T x$. (cf. Exercise 3)

Solution:
All pairs of distinct columns are linearly independent. The basic feasible solutions are

$v_1 = (0, 0, 3, 0, 0)^T$	bases: $\{1, 3\}, \{2, 3\}, \{3, 4\}, \{3, 5\}$
$v_2 = (0, 4, 0, 10, 0)^T$	bases: $\{2, 4\}$
$v_3 = (0, 0, 0, 18, 12)^T$	bases: $\{4, 5\}$
$v_4 = (9, 0, 0, 0, 30)^T$	bases: $\{1, 5\}$
$v_5 = (3, 6, 0, 0, 0)^T$	bases: $\{1, 2\}$

All other bases are infeasible. v_2 is the optimal solution. The objective function values of the other points can be computed easily.
b) All pairs of vertices are adjacent except of $\{v_2, v_4\}$ and $\{v_3, v_5\}$.
c)To sketch the given polyhedron you should draw in a 3-dimensional coordinate system all points according to the first three components, i.e.,

$$\begin{aligned} v_1^P &= (0, 0, 3)^T \\ v_2^P &= (0, 4, 0)^T \\ v_3^P &= (0, 0, 0)^T \\ v_4^P &= (9, 0, 0)^T \\ v_5^P &= (3, 6, 0)^T \end{aligned}$$

Now, connect each pair of adjacent vertices by a straight line edge. To find the vectors generating the cone of objective function vectors for which v_2^P is optimal we have to look at the active constraints and the corresponding vectors, i.e.,

$$\begin{aligned} -x_1 &\leq 0 & (-1,0,0)^T \\ -x_3 &\leq 0 & (0,0,-1)^T \\ -2x_1 + 3x_2 + 4x_3 &\leq 12 & (-2,3,4)^T \end{aligned}$$

The vectors given on the right generate the cone of objective function vectors for which v_2^P is an optimal solution.

Exercise 5 ($3 + 3^* + 2$ *Points*)
Let $\emptyset \neq X \subseteq \mathbb{R}^d$ be of finite cardinality and let $u, v, w \in \text{conv}(X)$ arbitrary.

a) Prove Caratheodory's Theorem which is

$$\exists x_0, \dots, x_d \in X.\ u \in \text{conv}\{x_0, \dots, x_d\}.$$

Solution:

In the following we will use X as index set for our variables.

The polyhedron

$$\{\lambda \in \mathbb{R}^X \mid \sum_{x \in X} \lambda_x x = u, \sum_{x \in X} \lambda_x = 1, \lambda \geq 0\}$$

is non-empty ($u \in \text{conv}(X)$) and bounded ($0 \leq \lambda \leq \mathbb{1}$). Thus[1], the polyhedron contains a basic feasible solution λ^*. Since there are $d+1$ equality constraints, at most $d+1$ entries of λ^* are nonzero. Hence, u is contained in the convex hull of at most $d+1$ points

$$\{x \in X \mid \lambda_x^* > 0\}.$$

b) Also prove the following extension of Caratheodory's Theorem. *(Bonus)*

$$\exists x_1, \dots, x_d \in X.\ u \in \text{conv}\{v, x_1, \dots, x_d\}.$$

Solution:

In the following we will use $X \cup \{v\}$ as index set for our variables.

The polyhedron

$$P = \left\{\lambda \in \mathbb{R}^{X \cup \{v\}} \mid \lambda_v v + \sum_{x \in X} \lambda_x x = u, \quad \lambda_v + \sum_{x \in X} \lambda_x = 1, \quad \lambda \geq 0\right\} = \{\lambda \mid A\lambda = b, \lambda \geq 0\}$$

is non-empty ($u \in \text{conv}(X)$) and bounded ($0 \leq \lambda \leq \mathbb{1}$). Thus, there is a feasible basis $B \subseteq X \cup \{v\}$ with $|B| \leq d+1$. If B contains v our claim is true. So assume B does not contains v.

Now, the idea is to do a single simplex step in order to bring v into the basis and move another point out of the basis.

Consider the following subsystem of $A\lambda = b$

$$A_B \lambda_B + A_v \lambda_v = b$$

[1]In the lecture we had the following theorem: Every non-empty polyhedron contains a basic feasible solution if and only if it does not contain a line. Clearly, bounded polyhedra cannot contain a line.

which is equivalent to

$$\lambda_B = A_B^{-1}b - A_B^{-1}A_v\lambda_v = \bar{b} - \bar{A}_v\lambda_v.$$

For all $\bar{\lambda}_v \geq 0$ such that $\bar{\lambda}_B = \bar{b} - \bar{A}_v\bar{\lambda}_v \geq 0$ we get a feasible solution $\bar{\lambda}$ of P by setting $\bar{\lambda}_x = 0$ for all $x \notin B \cup \{v\}$. Formally, we can write $\bar{\lambda} = \bar{\lambda}_B I_B + \bar{\lambda}_v I_v$.

Now, we show that we can find a λ'_v such that the corresponding λ'_B contains at most d positive entries which implies that $u \in \mathrm{conv}(\{v\} \cup \{x \in B \mid \lambda'_x > 0\})$.

Since P is bounded there must be a $\lambda'_v \geq 0$ such that the corresponding λ' is feasible but

$$\lambda_B = \bar{b} - \bar{A}_v\lambda_v \not\geq 0 \text{ for all } \lambda_v > \lambda'_v$$

Clearly, at least one component of $\lambda'_B := \bar{b} - \bar{A}_v\lambda'_v \geq 0$ must be zero. Thus, λ'_B has at most d nonzero entries and u is contained in the convex hull of at most $d+1$ points, namely

$$\{v\} \cup \{x \in B \mid \lambda'_x > 0\}.$$

c) Is the following statement true in general?

$$\exists x_2, \ldots, x_d \in X.\ u \in \mathrm{conv}\{v, w, x_2, \ldots, x_d\}.$$

Solution:

Consider a triangle T with vertices $\{a, b, c\} =: X$. Now, let v, w be two points in the interior of T. Clearly, the triangles $\mathrm{conv}(v, w, a), \mathrm{conv}(v, w, b), \mathrm{conv}(v, w, c)$ cannot cover $T = \mathrm{conv}(a, b, c)$. Any point u that is not covered is not contained in any convex hull $\mathrm{conv}(v, w, x)$ with $x \in X$.