Computational Game Theory

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2 Inefficiency of equilibria

The tragedy of the commons and the price of anarchy. We have seen in the preceding lectures how it is possible to model, using games and solution concepts, the kind of behavior that can arise in situations where multiple self-interested agents interact. What happens to the global behavior of the system? The concept of *social utility* can be used to measure the quality of a given state of a game.

Definition 2.1. The *social utility* of a game in state $s \in S$ is the quantity $\sum_{i \in N} u_i(s)$.

Remark 2.1. There are actually other possible definitions of social utility. The one we gave might be called *utilitarian* or *egalitarian*, as it gives the same weight to all players.

In general, the social utility that derives from the behavior at equilibrium is not the best possible one; the players of the game would globally be better off if a central optimal coordination was possible. The question is, how much worse than optimal can the social utility become at an equilibrium? This question is interesting because, if we knew that equilibria in a game have high social utility, comparable to the optimal one, then we can obtain more or less the same result as the optimal one without the need of enforcing the players to choose particular actions – which can often be expensive or impossible.

For example, consider the Bandwidth Sharing game. We have seen that the game has a pure Nash equilibrium where each of the n players uses a fraction $1/(n+1)$ of the bandwidth. The payoff for every player is then $1/(n+1)^2$, which means that the social utility at the equilibrium is $n/(n+1)^2 = \Theta(1/n)$. On the other hand, if every player used only a fraction $1/(2n)$ of the bandwidth, the payoff of each player would be $1/(4n)$, so the corresponding social utility would be $1/4$. Notice that is $\Theta(n)$ times larger than the social utility at the equilibrium. Thus, the price paid for the selfishness of the users is a dramatically decreased social utility. This well-known phenomenon is called the *tragedy of the commons* in Economics.

Although for some games the price of selfishness is high, this does not need hold in general. To quantify the degradation of the social utility, Koutsoupias and Papadimitriou have proposed the concept of *price of anarchy*, which is analogue to the notion of approximation ratio for optimization problems.

Definition 2.2. Let Γ be a normal-form game having a set of states S and let $E \subseteq S$ be a set of equilibria such that $E \neq \emptyset$. The *price of anarchy* of Γ (with respect to E) is the quantity

$$\frac{\max_{s \in S} \sum_{i \in N} u_i(s)}{\min_{s \in E} \sum_{i \in N} u_i(s)}.$$

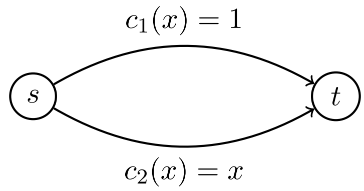

Figure 1: Pigou's network

When the game is defined in terms of costs $(c_i)_{i \in N}$, we instead use the definition

$$\frac{\max_{s \in E} \sum_{i \in N} c_i(s)}{\min_{s \in S} \sum_{i \in N} c_i(s)}.$$

Thus with this terminology the price of anarchy of the Bandwidth Sharing game (with respect to pure Nash equilibria) is $\Theta(n)$. Notice from the definition that the price of anarchy of any game is always at least 1. Notice also that in case the game admits multiple equilibria, the definition implicitly assumes that the worst one will occur (so that the guarantee always holds).

Exercise 2.1 (The pollution game)**.** In this game there are n countries (the players). Each country can decide of either passing the legislation to control pollution or not. Pollution control has a cost 3 for the country, while every country that pollutes adds 1 to the cost of *all* countries. Find the price of anarchy of the game.

3 Selfish routing

We have seen that the inefficiency of equilibria can be in general high and might not scale well with the dimension of the system being analyzed. However, it is possible to identify games for which equilibria are in fact approximately optimal; that is, games with bounded price of anarchy. We will now discuss a model of network games where this happens.

Pigou's example. Consider the simple network shown in Figure 1. Two disjoint arcs connect a source vertex s to a destination vertex t. Each arc is labeled with a *cost function* $c(\cdot)$ describing the cost (for example, the travel time) incurred by users of the arc, as a function of the amount of traffic routed on the arc. In the example, the upper arc has constant cost $c_1(x) = 1$. The cost of the lower arc is $c_2(x) = x$ and thus increases as the arc gets more congested. In fact, the lower arc is cheaper than the upper arc as long as less than one unit of traffic uses it.

Assume that one unit of traffic has to be routed in the network. This traffic is controlled by a very large population of players, each player controlling a negligible (infinitesimal) amount of flow from s to t. In fact, this is called a "nonatomic" selfish routing game because the decision of an individual player alone has no significant effect on the game. We assume that each player is minimizing its own cost, which is the sum of the costs of the arcs of the route he selects. Then we can see that the lower route is a dominant strategy for all players, so in the resulting dominating strategy equilibrium every player incurs one unit of cost. The social cost at the equilibrium is 1 (there is one unit of traffic, and all traffic incurs a cost of 1). In fact this is the only pure Nash equilibrium of the game.

What is the optimal social cost? If we send an amount of x ($0 \leq x \leq 1$) on the upper link, and $1 - x$ on the lower, we obtain a cost of 1 for the players using the upper link, and a cost of $1 - x$ for the players using the lower link. The social cost is thus

$$C(x) = x \cdot 1 + (1 - x) \cdot (1 - x) = 1 - x + x^2.$$

Since $C(x) = 2x - 1$ and $C''(x) > 0$, the social cost is minimized when $x = 1/2$, that is, when the traffic is evenly split. In this case we get a social cost of $3/4$. Thus the price of anarchy in this game (with respect to pure Nash equilibria) is equal to $4/3$.

What happens to the price of anarchy in more complex networks, when we increase the number arcs, or when there are multiple sources and destinations, or when we have non-linear latency functions? This is what we will study in the following.

3.1 The selfish routing model

A selfish routing game is specified by a *network* consisting of a directed graph $G = (V, A)$ with node set V and arc set A, together with a set $(s_1, t_1), \ldots, (s_k, t_k)$ of source-sink pairs (*commodities*). Each player carries an infinitesimal amount of flow associated with one commodity. We denote with $\mathcal{P}_i$ the set of s_i–t_i paths of the network. We define $\mathcal{P} := \cup_{i=1}^k \mathcal{P}_i$. We allow the graph to contain parallel arcs between the same pair of nodes.

A state of the game is represented by a *flow*, that is a function $f : \mathcal{P} \to \mathbb{R}_+$. If f is a flow and $P \in \mathcal{P}_i$, we denote with f_P the amount of traffic of commodity i that travels along the path from s_i to t_i. The game specifies a fixed *demand* $r_i \geq 0$ of traffic corresponding to each commodity i. A flow is *feasible* for a demand vector $r \in \mathbb{R}_+^k$ if for all commodities i, $\sum_{P \in \mathcal{P}_i} f_P = r_i$.

Each arc $a \in A$ has an associated *cost function* $c_a : \mathbb{R}_+ \to \mathbb{R}_+$. The cost functions are assumed to be nonnegative, continuous and nondecreasing. A *nonatomic selfish routing game* can thus be summarized by the triple (G, r, c), where G is the graph, r is the demand vector and c is the vector of cost functions.

What is the cost incurred by the players? If a player is routing along a path P and the global state of the game is represented by the flow f, the cost of the player is

$$c_P(f) := \sum_{a \in P} c_a(f_a)$$

where $f_a := \sum_{P \in \mathcal{P}: a \in P} f_P$ denotes the total amount of traffic using arc a.

Remark 3.1. Be careful not to confuse the two notations f_P and f_a. The first gives the flow traveling along a certain path P, without considering flows associated to other paths, even when they have some arc in common with the path P. The quantity f_a gives instead the total amount of flow traveling along a given arc a. We will avoid confusion by using capital letters for paths and small letters for arcs.

The social cost is measured as follows: the cost of a flow f is given by

$$C(f) := \sum_{P \in \mathcal{P}} f_P c_P(f).$$

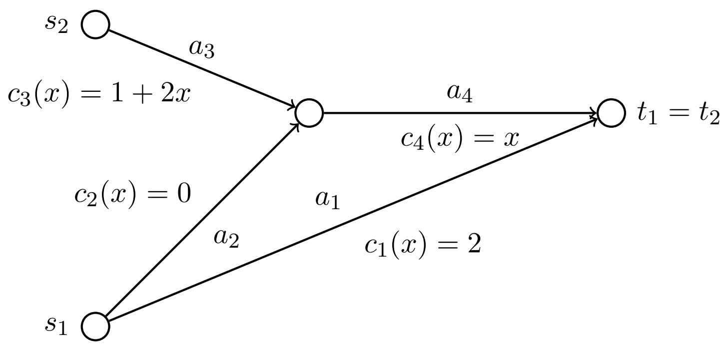

Figure 2: An example network

Equivalently (prove this), this can be expressed as the sum over all arcs

$$C(f) = \sum_{a \in A} f_a c_a(f_a).$$

Example 3.1. Consider the network in Figure 2. Here we assume two commodities ($k = 2$) with a demand of one each (demand vector $r = (1, 1)$). The source and destinations are (s_1, t_1) for the first commodity and (s_2, t_2) for the second. The graph contains two s_1–t_1 paths ($\mathcal{P}_1 = \{P_0, P_1\}$) and one s_2–t_2 path ($\mathcal{P}_2 = \{P_2\}$). The paths are the following: $P_0 = \{a_1\}$, $P_1 = \{a_2, a_4\}$, $P_2 = \{a_3, a_4\}$. Finally, the cost functions are $c_1(x) = 2$, $c_2(x) = 0$, $c_3(x) = 1 + 2x$, $c_4(x) = x$.

An example of feasible flow for the network is the flow f defined by $f_{P_0} = 1/2$, $f_{P_1} = 1/2$, $f_{P_2} = 1$. Notice that in this case $f_{a_1} = f_{P_0} = 1/2$, $f_{a_2} = f_{P_1} = 1/2$, $f_{a_3} = f_{P_2} = 1$, $f_{a_4} = f_{P_1} + f_{P_2} = 3/2$. The cost of this flow is thus $C(f) = (1/2) \cdot 2 + (1/2) \cdot 0 + 1 \cdot 3 + (3/2) \cdot (3/2) = 25/4$.

We can now define an equilibrium concept for nonatomic selfish routing games.

Definition 3.2. Let f be a feasible flow for the nonatomic instance (G, r, c). The flow f is an *equilibrium flow* if, for every commodities $i = 1, \ldots, k$ and every pair of paths $P, P' \in \mathcal{P}_i$,

$$\text{if } f_P > 0 \text{ then } c_P(f) \leq c_{P'}(f).$$

Notice that a consequence of the definition is that in an equilibrium flow, all nonzero s_i–t_i path flows f_P have the same cost, irrespective of P, for any $P \in \mathcal{P}_i$.

Example 3.3. In Pigou's example (Figure 1), the flow that sends all the traffic along the lower arc is an equilibrium flow.

Example 3.4. Consider again Example 3.1. The flow f mentioned in the example is not an equilibrium flow: consider the paths P_0 and P_1. We have $f_{P_0} > 0$ while $c_{P_0}(f) = 2 > c_{P_1}(f) = 3/2$. Thus, the players on the path P_0 have an incentive to change their route to P_1.

Instead, the flow g defined by $g_{P_0} = 0$, $g_{P_1} = 1$, $g_{P_2} = 1$ is an equilibrium flow (check the inequalities!).

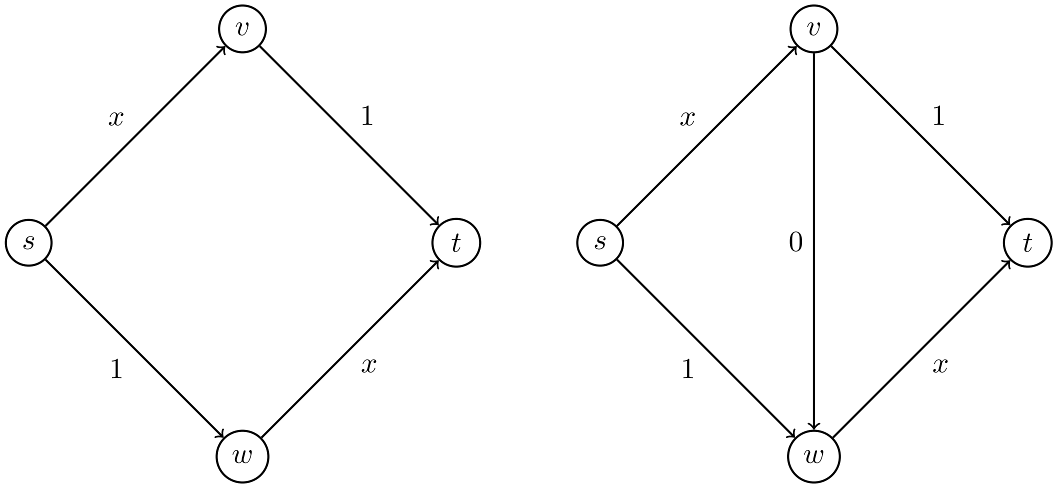

Figure 3: Braess' paradox

Example 3.5 (Braess' paradox)**.** The following example shows that in selfish routing games counter-intuitive phenomena can arise. Consider the network on the left of Figure 3. There are two routes, each with combined cost $1 + x$ when x traffic is sent along the route. Assume that there is a unit demand of traffic to be routed. Then the equilibrium flow is to split the traffic evenly (why?) and the total cost experienced by the traffic is $3/2$.

Now suppose that, in order to decrease the cost encountered by the traffic, we build a zero-cost arc connecting the midpoints of the existing routes, as on the right of Figure 3. What is the new equilibrium? There is a new route $s \to v \to w \to t$ and using this route is a (weakly) dominant strategy. Thus at equilibrium all the traffic is sent along this path. The total cost becomes thus 2, which is higher than before!

3.2 Existence of equilibrium flows

Our aim of this section is to show that a nonatomic selfish routing game always admits an equilibrium flow. We will make use of a characterization of *optimal* flows.

Definition 3.6. An *optimal flow* for instance (G, r, c) is a feasible flow f that minimizes $\sum_{a \in A} f_a c_a(f_a)$.

To simplify the discussion, assume that for any arc a, the function $x \cdot c_a(x)$ is continuosly differentiable and convex. Recall that $x \cdot c_a(x)$ is the contribution to the social cost of the traffic on arc a. Let $c^*_a(x) := (x \cdot c_a(x))' = c_a(x) + xc'_a(x)$ denote the *marginal cost function* for arc a. For example, if $c_a(x) = x^3$ then $c^*_a(x) = 4x^3$. Recall that with $c^*_P(f)$ we mean the quantity $\sum_{a \in P} c^*_a(f)$. Then it is possible to prove the following characterization (we omit the proof).

Proposition 3.1. *Under the above hypotheses, a feasible flow f^* is an optimal flow for (G, r, c) if and only if, for every commodity $i = 1, 2, \ldots, k$ and every pair of paths $P, P' \in \mathcal{P}_i$,*

$$\textit{if } f^*_P > 0 \textit{ then } c^*_P(f^*) \leq c^*_{P'}(f^*).$$

Notice the striking similarity between the above condition and the one in Definition 3.2! In fact, the conditions are the same, except that one is defined on the original costs c of the network, and the other on the marginal costs c^*. So we could say that optimal flows and equilibrium flows can be defined in the "same" way, but with respect to different cost functions.

Corollary 3.2. *A feasible flow is an optimal flow for (G, r, c) if and only if it is an equilibrium flow for (G, r, c^*).*

The idea now is the following: since optimal flows always exist (no matter which global function we use), we show that equilibrium flows exist by seeing them as optimal flows for a different notion of social cost. That is, we want to make a statement similar to Corollary 3.2, but in the "opposite" direction.

Let $h_a(x) := \int_0^x c_a(y)dy$. We make this choice because now $h'_a(x) = c_a(x)$. We also use the following definition.

Definition 3.7. The function

$$\Phi(f) := \sum_{a \in A} \int_0^{f_a} c_a(x)dx = \sum_{a \in A} h_a(f_a)$$

is called the *potential function* of a nonatomic instance (G, r, c).

We can now invoke Proposition 3.1, except that we consider the minimization of $\Phi(\cdot)$ instead of the minimization of $C(\cdot)$; that is, we use the function $h_a(x)$ in place of $x \cdot c_a(x)$. Then what was before $c^*_a(x) = (x \cdot c_a(x))'$ is now $h'_a(x) = c_a(x)$, so that the second part of Proposition 3.1 is stating that we have an equilibrium flow with respect to costs $(c_a)_{a \in A}$. We have reformulated Proposition 3.1 as follows.

Proposition 3.3. *A feasible flow is an equilibrium flow for (G, r, c) if and only if it is a global minimum of the corresponding potential function Φ given in Definition 3.7.*

Armed with this proposition we can now prove our main existence result.

Theorem 3.4. *Let (G, r, c) be a nonatomic instance. Then*

1. *The instance (G, r, c) admits at least one equilibrium flow.*

2. *If f and g are equilibrium flows for (G, r, c) then $c_a(f_a) = c_a(g_a)$ for every arc a.*

Proof. By its definition, the set of feasible flows is a compact (= closed and bounded) subset of $\mathbb{R}^{|\mathcal{P}|}$. Arc cost functions are continuous, so the potential function is also continuous. By Weierstrass' Theorem from elementary analysis (remember?), Φ achieves a minimum on this set. By Proposition 3.3, every one of this minima corresponds to an equilibrium flow of (G, r, c). This proves (1).

Part (2) can be proved by using the fact that f and g are both minimizers of Φ and the fact that Φ is convex, thus it must be constant between f and g. Moreover Φ is the sum of convex terms, so every term $\int_0^x c_a(x)dx$ must be linear between f and g, which implies that c_a is constant between f and g. $\square$

Example 3.8. We can use Corollary 3.2 to find an optimal flow for the Braess network (right part of Figure 3). If we compute the marginal cost functions, we obtain $(x \cdot x)' = 2x$ for the arcs that had cost x, while the arcs with constant cost (0 and 1) remain unaltered (because for example $(x \cdot 1)' = 1$). If we search an equilibrium flow in the network with modified costs, we obtain the flow that sends half unit of traffic on the upper path, half unit on the lower path, and no unit on the zig-zag path. By Corollary 3.2 this flow is the optimal flow for the network with the original cost functions. It has cost 3/2. On the other hand, we saw in Example 3.5 that the equilibrium flow has cost 2. So the price of anarchy for the Braess graph is 4/3 (as it was for the Pigou network).

4 The price of anarchy for selfish routing

4.1 Bound using the potential function

Having proved in the previous section that an equilibrium flow always exists, we can now analyze the price of anarchy of selfish routing.

We first show that the type of cost functions plays an important role.

Example 4.1 (Nonlinear Pigou)**.** Consider again Pigou's network, except that now the linear cost function $c_2(x) = x$ is replaced by the quadratic function $c_2(x) = x^2$.

It is easy to see that once more, an equilibrium flow sends all the demand (which is equal to 1) on the lower link, for a social cost of 1. What is the optimal flow? If we send an amount of x $(0 \leq x \leq 1)$ on the upper link, and $1 - x$ on the lower, we obtain a cost of 1 for the players using the upper link, and a cost of $(1 - x)^2$ for the players using the lower link. The social cost is thus

$$C(x) = x \cdot 1 + (1 - x) \cdot (1 - x)^2 = x + (1 - x)^3.$$

As $C'(x) = 1 - 3(1 - x)^2$, we obtain the minimum when $(1 - x)^2 = 1/3$, that is when $x = 1 - 1/\sqrt{3}$. The cost in this case is $1 - (2/3)(1/\sqrt{3}) \simeq 0.615$. Thus the price of anarchy is $\simeq 1/0.615 \simeq 1.626$. Notice that this is larger than the 4/3 we obtained with a linear cost function.

In fact, if we generalize the example with $c_2(x) = x^p$, we obtain a price of anarchy growing roughly as $p/\ln p$. Thus the price of anarchy is not bounded if we do not limit the class of cost functions.

The example shows that if the cost functions can be "highly nonlinear", the price of anarchy can be very high even on very simple networks. But what if the cost functions are for example linear or quadratic? Can we obtain a useful bound in this case?

Theorem 4.1. *Suppose that $x \cdot c_a(x) \leq \gamma \cdot \int_0^x c_a(y)dy$ for all $a \in A$ and $x \geq 0$. Then the price of anarchy of (G, r, c) is at most γ.*

Proof. Let f be an equilibrium flow and let f^* be an optimal flow. Since cost functions are non-decreasing, the cost of a flow is always at least its potential function value (why?), in particular $C(f^*) \geq \Phi(f^*)$. By the hypothesis, the cost of any flow is at most γ times the potential value of the flow. We can conclude, using Proposition 3.3,

$$C(f) \leq \gamma \cdot \Phi(f) \leq \gamma \cdot \Phi(f^*) \leq \gamma \cdot C(f^*).$$

□

Corollary 4.2. *The price of anarchy in nonatomic instances with cost functions that are polynomials of degree at most p (with nonnegative coefficients) is at most $p+1$.*

Proof. For any arc a, there exist nonnegative coefficients $b_0, \ldots, b_p$ such that

$$c_a(x) = b_p x^p + b_{p-1} x^{p-1} + \ldots + b_0.$$

Then

$$x \cdot c_a(x) = b_p x^{p+1} + b_{p-1} x^p + \ldots + b_0 x.$$

On the other hand,

$$\int_0^x c_a(y)dy = \frac{1}{p+1} b_p x^{p+1} + \frac{1}{p} b_{p-1} x^p + \ldots + b_0 x.$$

By direct comparison we conclude that

$$x \cdot c_a(x) \leq (p+1) \int_0^x c_a(y)dy,$$

so by Theorem 4.1 the price of anarchy is at most $p+1$. □

4.2 Bound using the variational inequality

In this section we improve the upper bound for *affine* cost functions (with nonnegative coefficients), that is, functions of the form $c_a(x_a) = b_{1a} x_a + b_{0a}$ where $b_{0a}, b_{1a} \geq 0$. In this case Corollary 4.2 gives an upper bound of 2 on the price of anarchy. We will show an improved upper bound of 4/3. Since this matches the lower bound from Pigou's example, this improved upper bound is best possible.

We will use another characterization of equilibrium flows, given by the following proposition.

Proposition 4.3. *Let f be a feasible flow for (G, r, c). Then f is an equilibrium flow if and only if*

$$\sum_{a \in A} f_a c_a(f_a) \leq \sum_{a \in A} g_a c_a(f_a) \tag{1}$$

for every flow g feasible for (G, r, c).

Proof. Fix a flow f and define the function H_f on the set of feasible flows as follows:

$$H_f(g) = \sum_{i=1}^{k} \sum_{P \in \mathcal{P}_i} g_P c_P(f) = \sum_{a \in A} g_a c_a(f_a).$$

The value $H_f(g)$ denotes the cost of a flow g after the cost function of each arc a has been changed to the constant function everywhere equal to $c_a(f_a)$. By the second definition of H_f, the claim we have to prove is equivalent to the assertion that a flow f is an equilibrium flow if and only if it minimizes the function $H_f(\cdot)$ over all feasible flows.

Consider now the first definition of H_f. From this we see that a flow g minimizes H_f if and only if it is a minimum cost flow with respect to the constant cost functions $c_a(f_a)$. That is, it is the optimum of a minimum cost flow problem without any capacity constraints or integrality constraints. Such a flow g is optimum if and only if it only sends flow along paths of minimum cost, that is $g_P > 0$ only for paths P that minimize $c_P(f)$ over all s_i–t_i paths. But if f is an equilibrium flow and $g = f$, this is clearly satisfied. So every equilibrium flow f satisfies (1). On the other hand, suppose f is not an equilibrium flow. Then there is a commodity i and two paths $P, P' \in \mathcal{P}_i$ such that

$$f_P > 0 \text{ and } c_P(f) > c_{P'}(f).$$

Now take g to be the same as f except that the flow on path P is moved to path P'. Since the cost is strictly smaller, we obtain

$$\sum_{i=1}^{k} \sum_{P \in \mathcal{P}_i} g_P c_P(f) < \sum_{i=1}^{k} \sum_{P \in \mathcal{P}_i} f_P c_P(f),$$

in other words, (1) is not satisfied. Thus (1) is satisfied for all feasible g if and only if f is an equilibrium flow.

Remark. Observe that (1) can be written as

$$(f - g) \cdot \nabla\Phi(f) \leq 0,$$

which explains the name *variational inequality.* This kind of inequality is common in optimization, economics and physics. □

We can now prove our improved bound (note: the proof given here is easier than the one in the book by Nisan et al.).

Theorem 4.4. *Let f be an equilibrium flow of instance (G, r, c) where the functions $(c_a)_{a\in A}$ are affine, and let f^* be an optimal flow for the same instance. Then $C(f) \leq (4/3)C(f^*)$.*

Proof. Thanks to Proposition 4.3, we have

$$\begin{aligned} C(f) &= \sum_{a\in A} f_a c_a(f_a) \\ &\leq \sum_{a\in A} f_a^* c_a(f_a) = \\ &= \sum_{a\in A} f_a^* c_a(f_a^*) + \sum_{a\in A} f_a^*(c_a(f_a) - c_a(f_a^*)) = \\ &= C(f^*) + \sum_{a\in A} f_a^*(c_a(f_a) - c_a(f_a^*)). \end{aligned}$$

For the arcs where $f_a^* \geq f_a$, this bound is already sufficient because the cost functions are nondecreasing and so the second term of the sum becomes negative. So it is enough to focus on arcs for which $f_a^* < f_a$. In this case, $f_a^*(c_a(f_a) - c_a(f_a^*))$ is equal to the area of the shaded rectangle in Figure 4. Note that the area of any rectangle whose upper-left corner point is $(0, c_a(f_a))$ and whose lower-right corner point

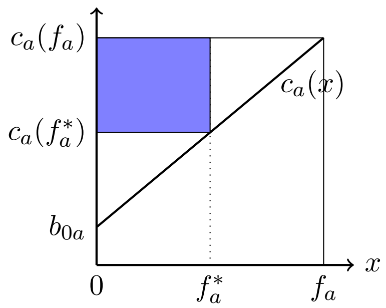

Figure 4: Illustration of the proof of Theorem 4.4

lies on the line representing $c_a(y) = b_{1a}y_a + b_{0a}$, is at most half of the triangle defined by the three points $(0, c_a(f_a))$, $(0, b_{0a})$ and $(f_a, c_a(f_a))$. In particular,

$$f_a^*(c_a(f_a) - c_a(f_a^*)) \leq \frac{1}{4} f_a c_a(f_a).$$

This completes the proof, as we obtain

$$C(f) \leq C(f^*) + \frac{1}{4}C(f),$$

that is, $C(f) \leq (4/3)C(f^*)$. □

4.3 Bicriteria bound for general cost functions

As we have seen, for general cost functions it is not possible to give a bound on the price of anarchy, not even on very simple networks. However one can prove the following.

Theorem 4.5. *If f is an equilibrium flow for (G, r, c) and f^* is any feasible flow for $(G, 2r, c)$, then $C(f) \leq C(f^*)$.*

That is, the equilibrium flow has better cost than any flow that has to send *twice as much* traffic (including the optimal flow for twice the traffic). Intuitively, this means that the inefficiency due to selfish routing can be repaired by using links that have twice the “capacity” of the original ones.

Proof. Let f and f^* be as in the statement of the theorem. For a commodity i, let d_i denote the minimum cost of a s_i–t_i path with respect to the flow f, so that $C(f) = \sum_i r_i d_i$.

Define a new set of cost functions $\bar{c}$ as follows: $\bar{c}_a(x) = \max\{c_a(f_a), c_a(x)\}$ for each $a \in A$. Let $\bar{C}(\cdot)$ denote the cost of a flow in the instance $(G, r, \bar{c})$. Observe that $\bar{C}(f^*) \geq C(f^*)$ and $\bar{C}(f) = C(f)$.

Now for every arc a, $\bar{c}_a(x) - c_a(x)$ is zero for $x \geq f_a$ and bounded above by $c_a(f_a)$ for $x < f_a$, so $x(\bar{c}_a(x) - c_a(x)) \leq c_a(f_a) f_a$ for all $x \geq 0$. Thus

$$\bar{C}(f^*) - C(f^*) = \sum_a f_a^*(\bar{c}_a(f_a^*) - c_a(f_a^*)) \leq \sum_a c_a(f_a) f_a = C(f).$$

On the other hand, the modified cost $\bar{c}_a(\cdot)$ is always at least $c_a(f_a)$, so the cost $\bar{c}_P(\cdot)$ of a path $P \in \mathcal{P}_i$ is always at least $c_P(f) \geq d_i$. We obtain

$$\bar{C}(f^*) = \sum_{P \in \mathcal{P}} \bar{c}_P(f^*) f_P^* \geq \sum_{i=1}^{k} \sum_{P \in \mathcal{P}_i} d_i f_P^* = \sum_{i=1}^{k} 2 r_i d_i = 2C(f).$$

Comparing the two inequalities we obtain

$$2C(f) \leq \bar{C}(f^*) \leq C(f) + C(f^*),$$

so that $C(f) \leq C(f^*)$, as claimed. □