Divided-and-Conquer for Voronoi Diagrams Revisited

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Outline

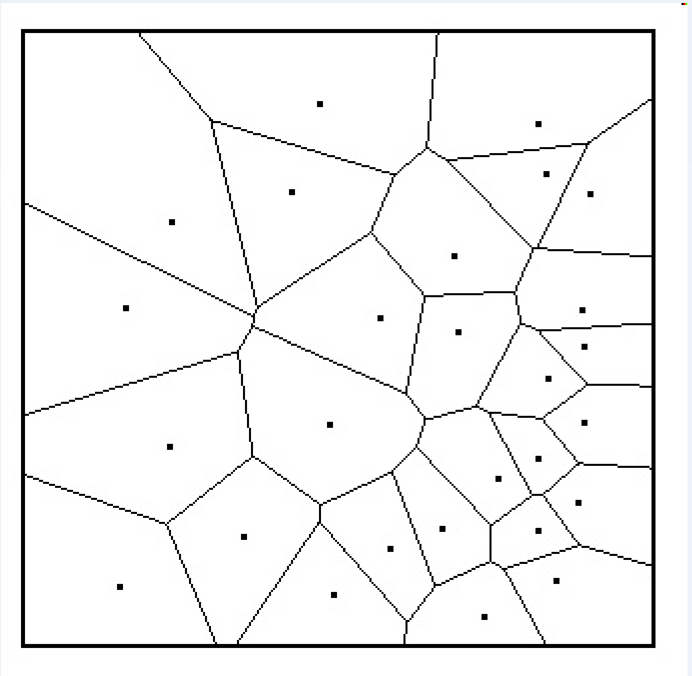
- Introduction
- Generalized Voronoi Diagram
- Algorithm for building generalized Voronoi Diagram
- Applications

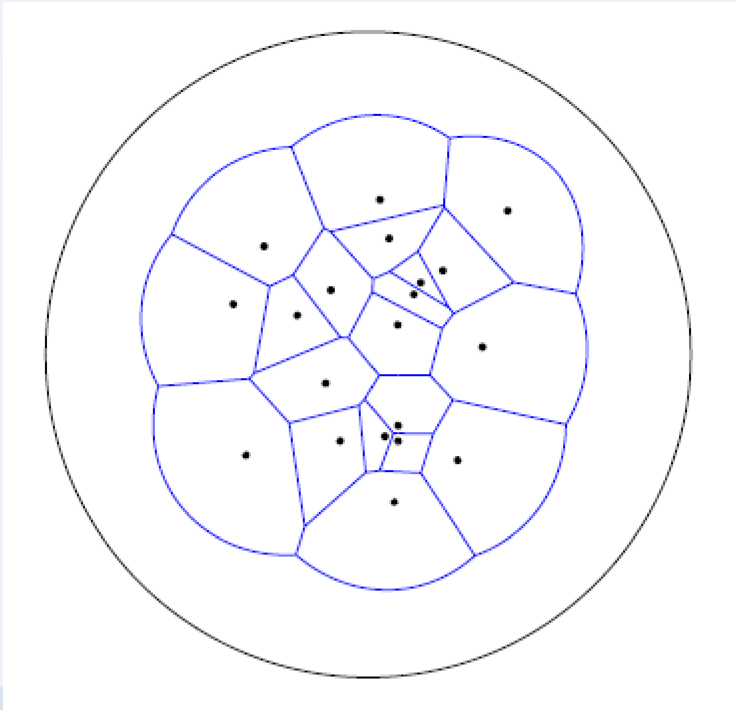
Introduction

- Voronoi Diagram with point sites
- Divide-and-conquer algorithm
- Generalized Voronoi Diagram
- Medial Axis and Medial Axis Transformation
- What do we want to do here?

Voronoi Diagram

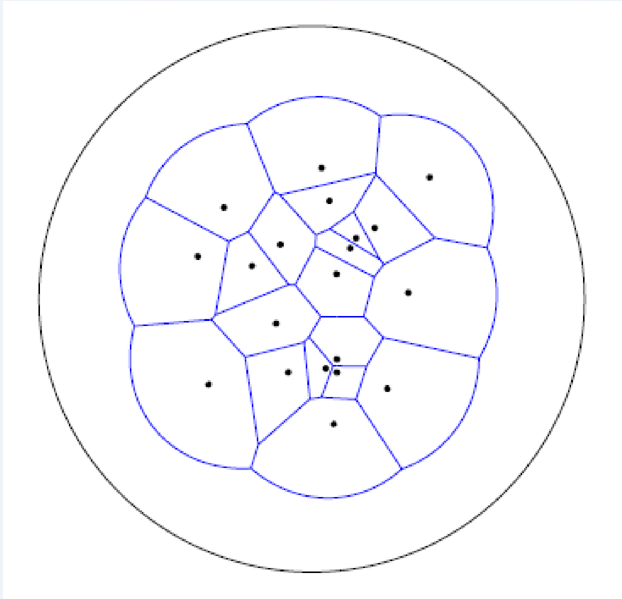
- Definition
  - In mathematics, a Voronoi diagram is a special kind of decomposition of a metric space determined by distances to a specified discrete set of objects in the space, e.g., by a discrete set of points.





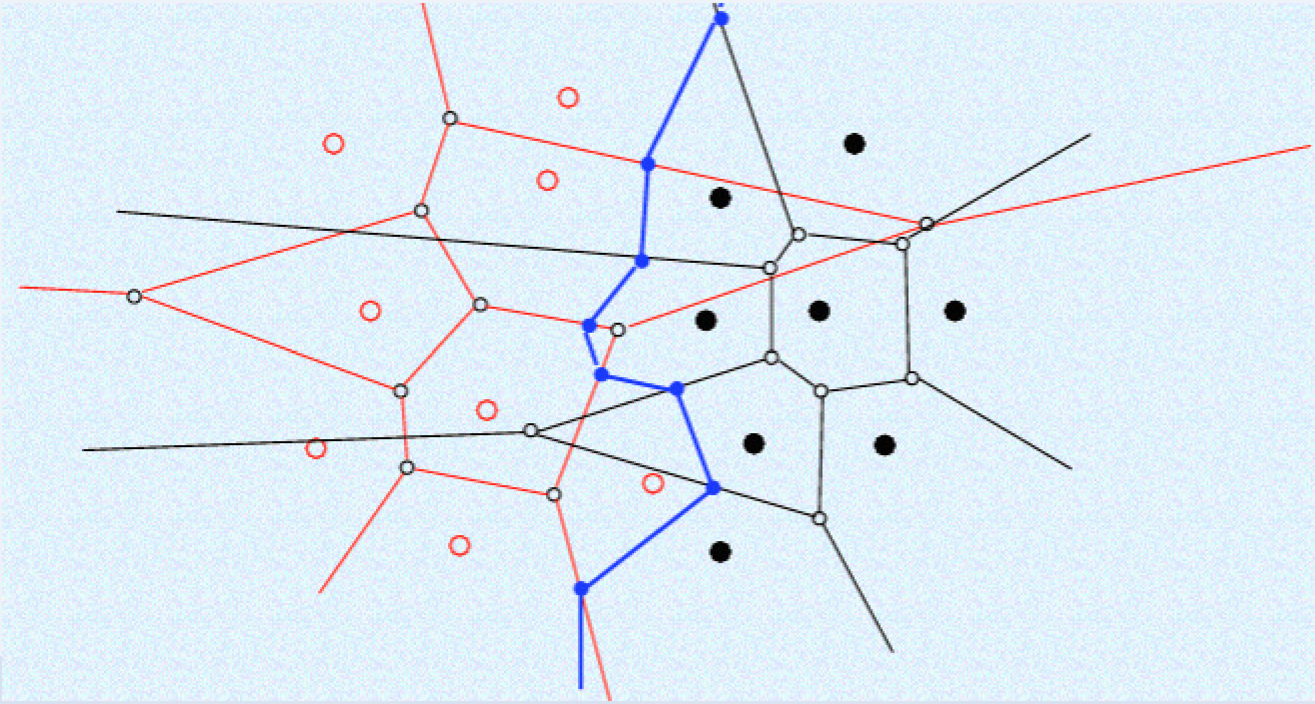
Voronoi Diagram

- Site
- Edge
  - Set of points each of those has eqaul distance to the sites which regions are using this edge.
- Vertex
  - Intersecting points of edges
- Region
  - Open area inside bounding edges
    - homomephic to open disk
- Bounding cricle
- Properties
  - Equidistance
  - Closeness
  - Bijection (site, region)



Divide-and-conquer algorithm

- Divide and conquer algorithm
  - The original problem is recursively divided into several simpler sub-problems of roughly equal size, and the solution of the original problem obtained by merging the solutions of the sub-problems.

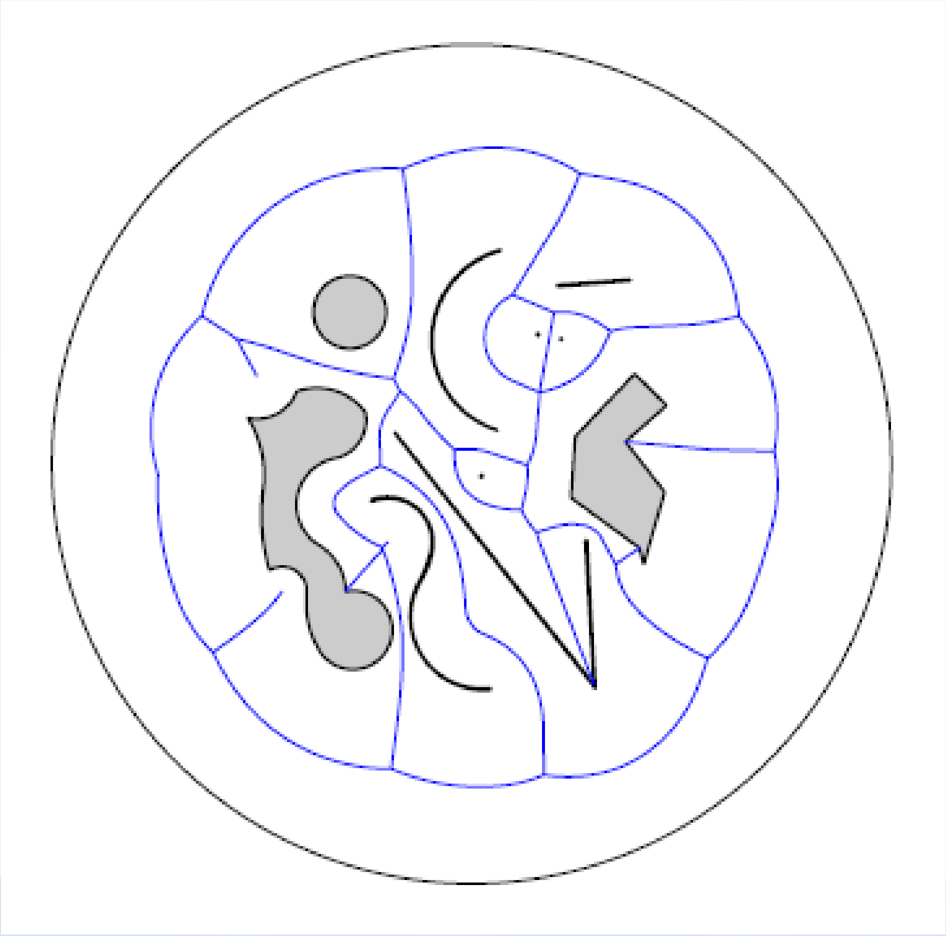


Divide-and-conquer algorithm

- Example
  - http://www.personal.kent.edu/~rmuhamma/Compgeometry/MyCG/Voronoi/DivConqVor/divConqVor.htm

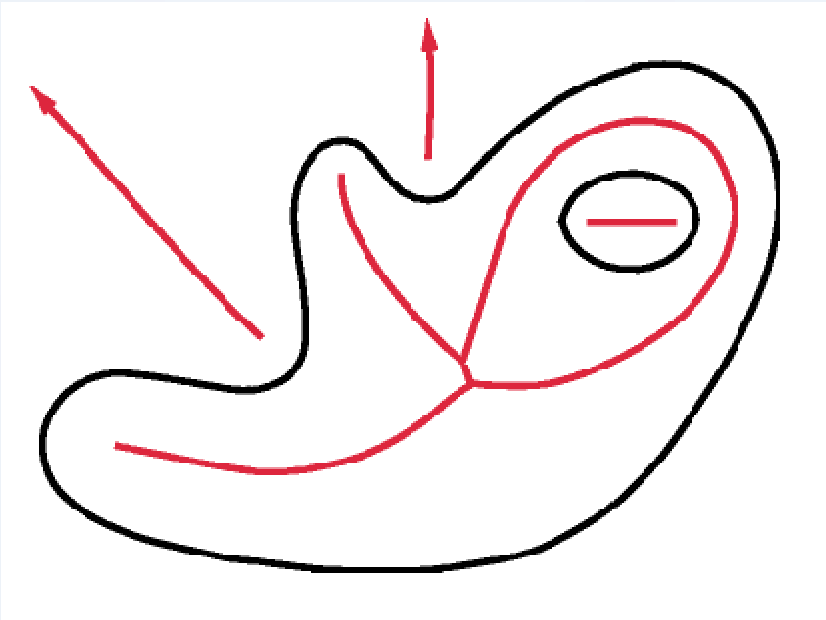
Generalized Voronoi Diagram

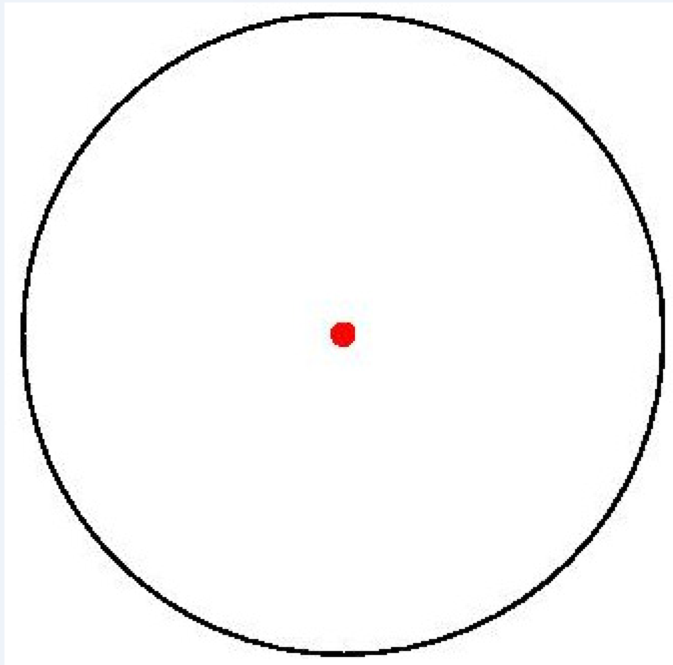
- When point sites becomes shapes in 2D
  - Points
  - Arcs
  - Closed planar areas
- Differences
  - Sites
  - Edges
  - Verticies
  - Regions



Medial Axis (MA) and Transformation (MAT)

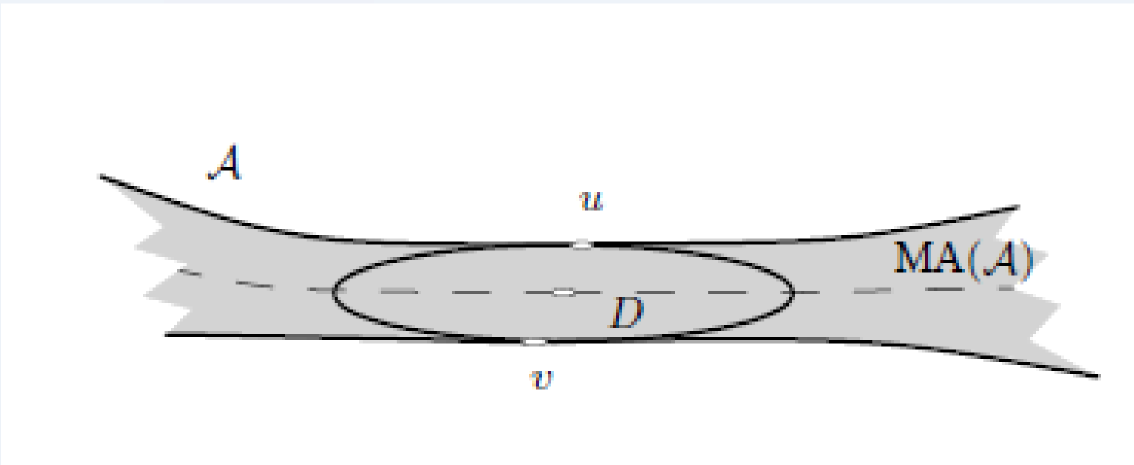
- Medial Axis (MA)
  - The medial axis of an object is the set of all points having more than one closest point on the object's boundary.





Medial Axis (MA) and Transformation (MAT)

- Medial Axis Transformation (MAT)
  - The medial axis together with the associated radius function of the maximally inscribed discs
  - The medial axis transform is a complete shape descriptor (see also shape analysis), meaning that it can be used to reconstruct the shape of the original domain.



What do we want to do here?

- Generate generalized voronoi diagram for 2D sites in plane

Generalized Voronoi Diagrams

- Definition
  - The voronoi diagram for general sites, V(S), of set of sites S is edge graph, Gs, which is the set of all points having more than one closest point on the union of all sites.
- Sites
  - Pairwise disjoint, closed topological disks of dimension two, one or zero in Euclidean plane **R^2**.
  - Dimension two: object homeomorphic to a disk
  - Dimension one: object homeomorphic to a line segment
  - Dimension zero: point
- Distance of a point x to a site s $\in$ S (δ is Euclidean distance func.)

$$d(x, s) = \min_{y \in s} \delta(x, y)$$

Generalized Voronoi Diagram

- Dividing the Voronoi Diagram
- Augmented domains

Dividing the Voronoi Diagram

- Observation

  1. The region of V(S) bijectively correspond to the sites in S.

  2. Each site is contained in its region. (bijection)

  3. Regions are simply connected.

  Proof: Assume that **x** is a point in region **R** of **V(S)**, To **x** there exists a unique closest point **y**, on the union of the sites in **S** (If **x** is not on edge of **Gs**).

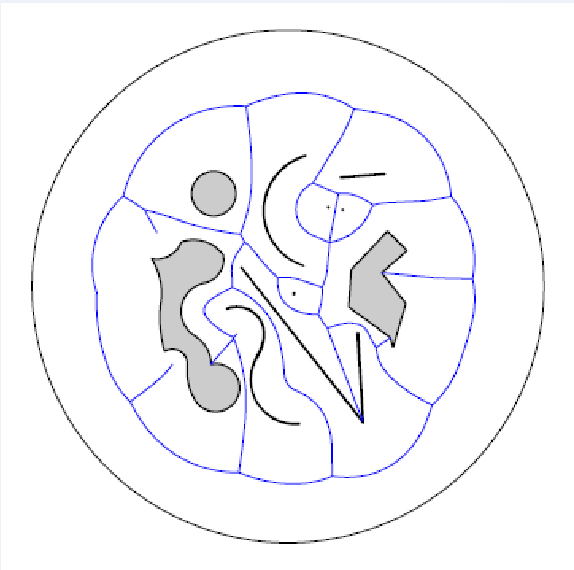
  Sites are pairwise disjoint, so there is a unique site **s** $\in$ S with **y** $\in$ **s**. Site **s** is the same for all **x** $\in$ R, since **d (x, s)** is a continuous function of **x**. This maps regions to sites.

  With **x** also the closed line segment **xy** is part of R. This imples that R is simply connected.

  y $\in$ R imples **s** $\subset$ R and maps sites to regions.

Dividing the Voronoi Diagram

- Bounds
  - Surrounding circle Γ : introduce surrounding circle Γ into sites **S**
  - We choose Γ s.t. each vertex of **V(S \ Γ)** is also a vertex of **V(S)**
  - All regions of **V(S)** are bounded now, except **R (Γ).**



Dividing the Voronoi Diagram

- Break the edge graph **Gs**
  - Let **p(s)** be a point on **s** with smallest ordinate (y-coordinate), and **q(s)** is the point on **Gs** vertically below **p(s).**
  - Since Region for **S** Γ is bounded, so **q(s)** always exists.
  - We assume that **q(s)** is not an endpoint of any edge of **Gs**. If it is the not the case, we rotate coordinate system slightly.
  - Wd define **Ts**, as follows (remove all **q(s)** from **Gs**):

$$\mathcal{T}_S = \mathcal{G}_S \setminus \{q(s) \mid s \in S \setminus \{\Gamma\}\}$$

Dividing the Voronoi Diagram

- Lemma 1. The graph Ts is a tree

proof (sketch):

every region is closed => **q (s)** always exists **(acyclic)**

bijection of site and region => every cycle of region is only one point missing, that is **q(s).**

Assume adjacent points of **q(s)** are **ql(s)** and **qr(s)** => we can always travel from **ql(s)** to **qr(s)** along the remained region cycle

q(s) is not vertex of **V(S)** => we could travel from **ql(s)** or **qr(s)** to any point of **V(S)** **(reachability).**

So, **acyclic** and **reachability** => **Ts** is a tree.

Augmented Domains

- We interpret Ts as medial axis of a generalized planar domain.
- To construct V(S) is identical to construct medial axis and using the medial axis algorithm.

Augmented Domains

- Shape: two-manifold B, which contains objects in plane or sphere surface of ball.
- Inscribed disk for B: disk lies entirely in B.
- Medial axis transformation (MAT(B)): set of all maximal inscribed disks.
- Medial axis (MA(B)): set of centers of maximal inscribed disks.
- We interpret V(S) as MA of a planar shape: simply take surrounding circle Γ as part of shape boundray, and consider each remained site $s \in S$ as a hole.
- It is like that we have a disk $\mathbf{B_0}$ in plane surrounded by Γ, and make some holes on it, which are sites $s \in S$. The shape is the remained disk after making holes. So we define:

$$\mathcal{B} = B_0 \setminus \{s \in S \mid s \neq \Gamma\}$$

Augmented Domains

- We disconnect shape B at appropriate positions, s.t. medial axis of resulting domain corresponds to the tree decomposition Ts of V(S).
- Introduce augmented domain and construct it recursively.

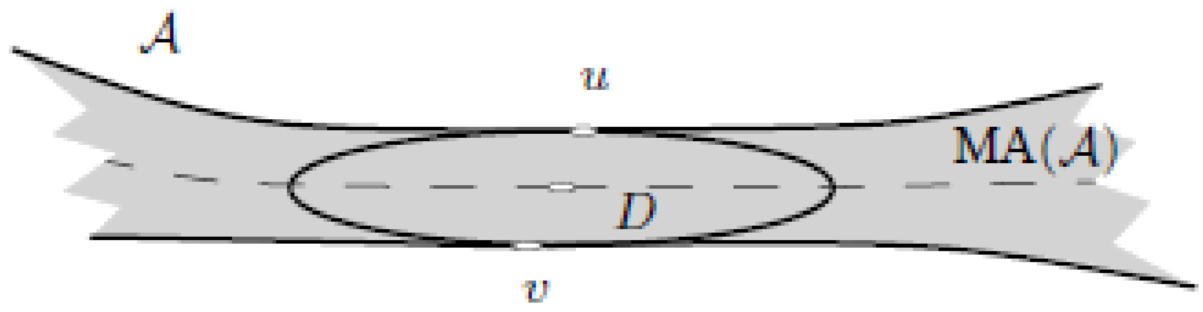
Augmented Domains

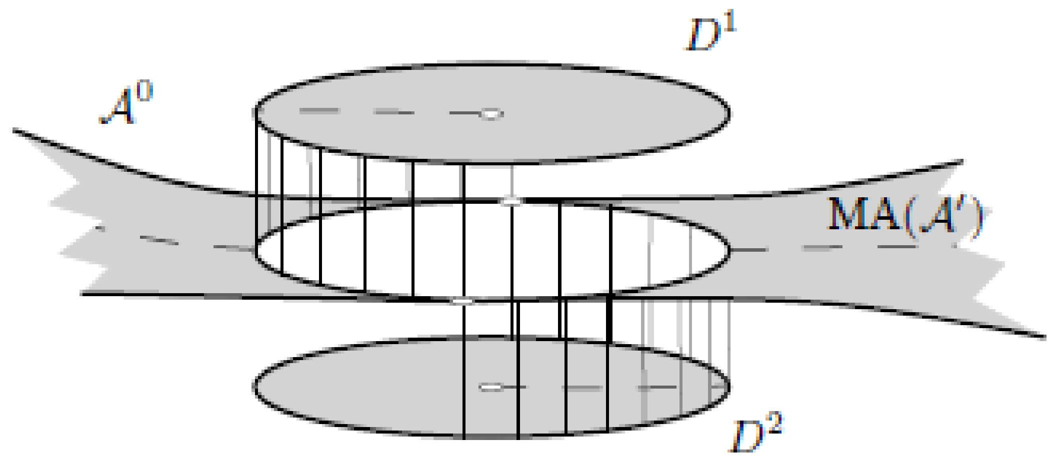
- Augmented domain: set **A** together with a projection π_{A} : **A -> R²**.
- Initially, A is original shape B, and π_{A} is identical to **A**.
- Consider a maximal inscribed disk **D** of **A**, which touches the boundray ∂**A** in exactly two points u and v. Namely, we split boundray of **D** into **uv** and vu, two circular arcs.
- **A'** is new augmented shape (lifting **D** to **D¹** and **D²**):

$$\mathcal{A}' = \mathcal{A}^0 \cup D^1 \cup D^2$$

$$\mathcal{A}^0 = \{(x,0) \mid x \in A \setminus D\}$$
$$D^1 = \{(x,1) \mid x \in D\}$$
$$D^2 = \{(x,2) \mid x \in D\}$$

Augmented Domains





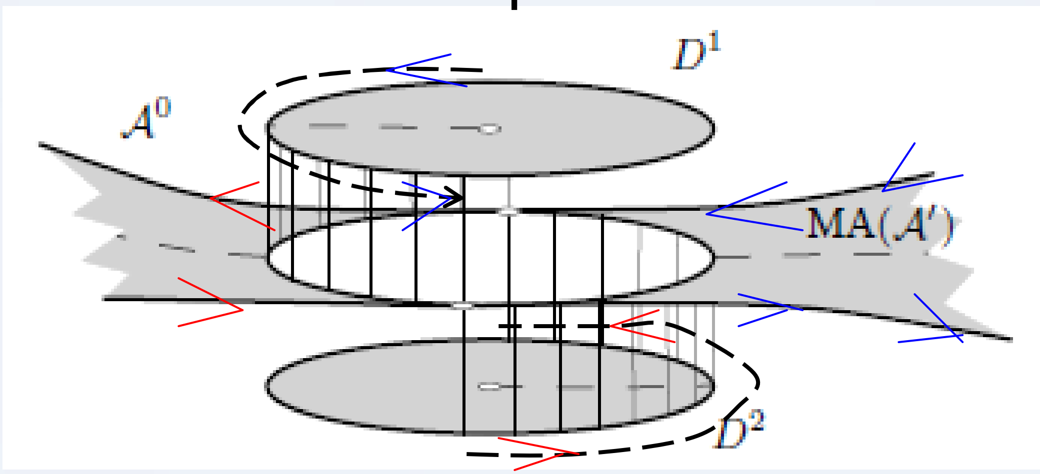
$$\pi_{\mathcal{A}'} : \mathcal{A}' \to \mathbb{R}^2,\ (x,i) \mapsto \pi_{\mathcal{A}}(x)$$

Augmented Domains

- Line segment ((x, i) , (y, j)) in A is contained in A' if one of the following conditions is met:
  - **i = j** and line segment **xy** avoids ∂**D**;
  - **{i, j} = {0,1}** and **xy** intersects the arc **uv**;
  - **{i, j} = {0, 2}** and **xy** intersects the arc **vu**;
- The distance of two points (x, i), (y, j) in A' is distance of $\pi_A(x)$ and $\pi_A(y)$ in $\mathbf{R}^2$ if line segment is contained in A; the distance is ∞ if line segment is not contained in A.
- An open disk in A' centered at (m, i) with radiu δ is the set of all points in A' whose distance to (m, i) is less than δ. It is inscribed in A' if its projection into R^2 is again an open disk.
- Boundray of A', ∂A', can be gotten by disconnecting the A's boundray at the contact points u and v of the splitting disk D, and reconnecting it with the circular arcs uv and vu.

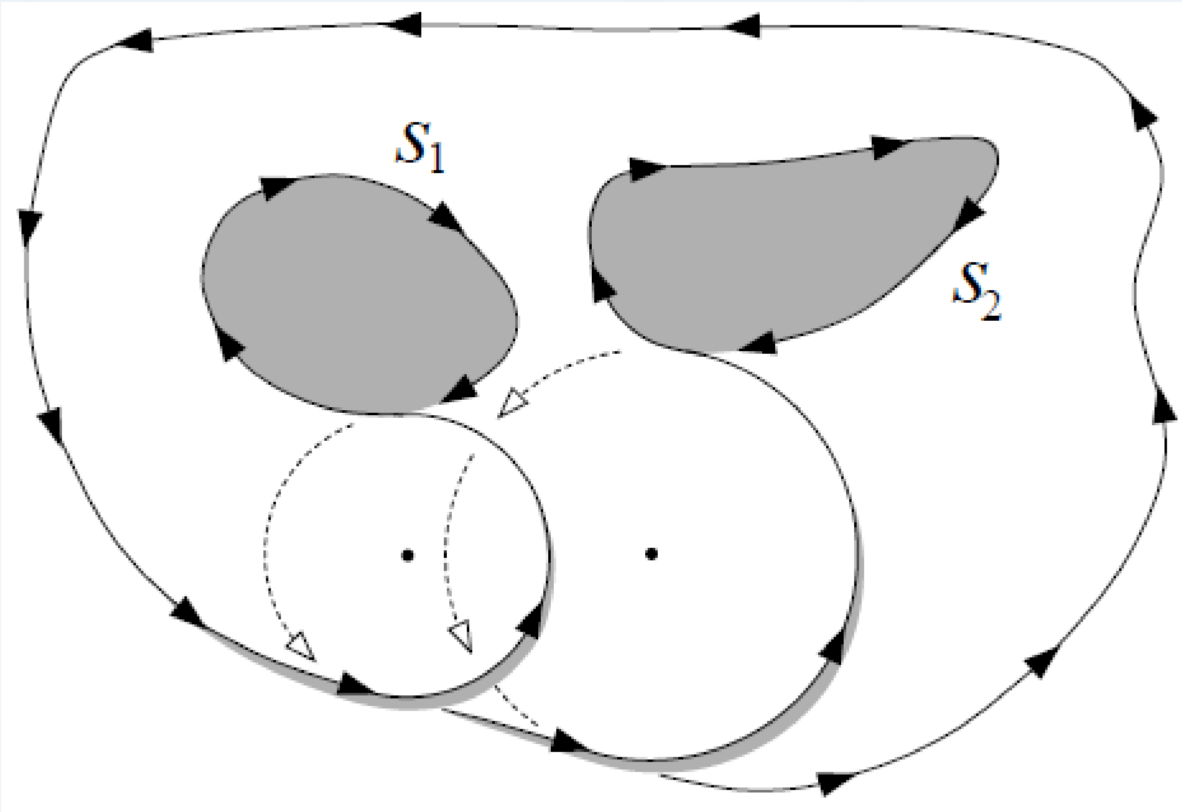
Augmented Domains

S_i



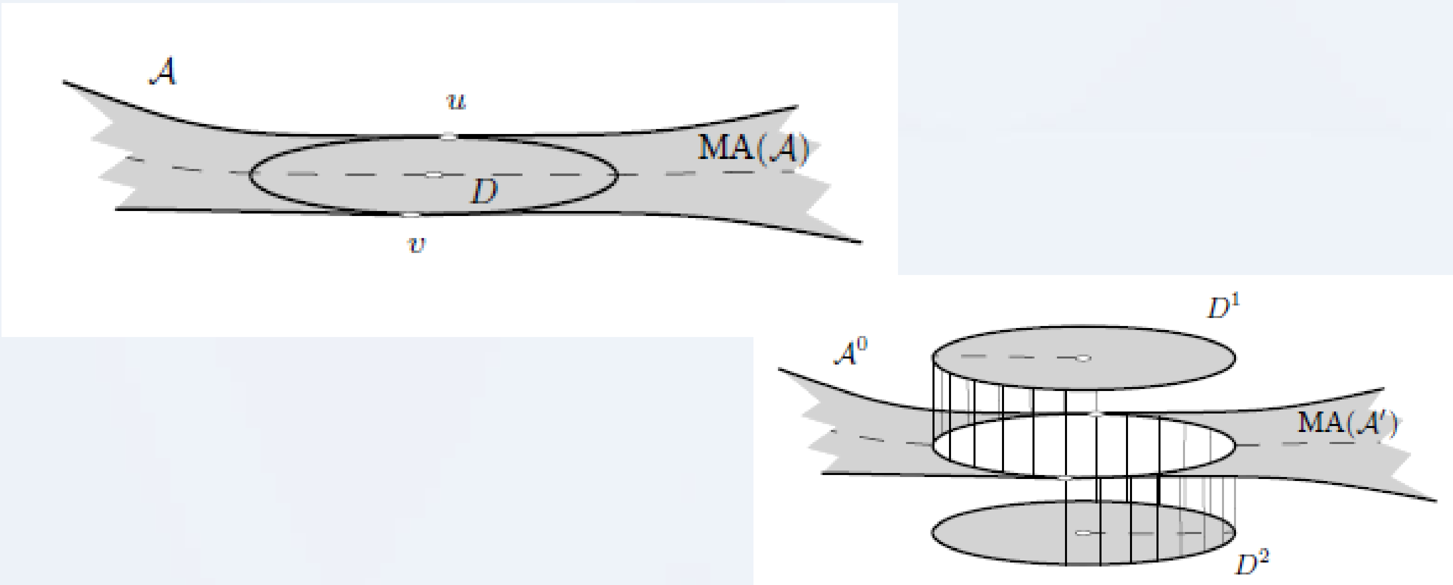
Augmented Domains

- We have made boundray directional and travesable.



Augmented Domains

- Bijection relation between MAT(A)\ {D} and MAT(A')\{D^1, D^2}.
- MA(A') is the same as MA(A), only except the edge in MA(A) containing the center of D which is splitted into two disconnected edges which both have center of D.



Augmented Domains

- We intend to draw edge graph Gs. And we do it like follows:
  - Initial shape B is augmented with |S| - 1 maximal inscribed disks,
  - The disks are centered at the points q(s) of Gs, q(s) was the vertical projection onto Gs of a point with smallest ordinate on the site s.

- After these |S| - 1 augmentations, denote current domain as As.

- Lemma2. The tree Ts is the MA of the augmented domain As.

Augmented Domains

- The boundray acturally determines the direction of our travel.
- It makes us to travel from one point on boundray and along the boundray, we can finally return to the same point.

Algorithm for building generalized Voronoi Diagram

- Computing boundary of augmented domain As
- Computing medial axis of augmented domain As
- And the medial axis is Gs of V(S)

- Experiments

Algorithm for building generalized Voronoi Diagram

- To construct the boundray of As is to find the maximal inscribed disks for all sites in S\ Γ .

- To find the disks is to find the centers and redii of disks.

Algorithm for building generalized Voronoi Diagram

- Recall that $\mathbf{D_i}$ is horizontally tangent to $\mathbf{s_i}$ at a lowest point $\mathbf{p(s_i)}$ of $\mathbf{s_i}$.

- The center $\mathbf{q(s_i)}$ of $\mathbf{D_i}$ lies on edge graph **Gs** of **V(S)**.

- So, we derive our main algorithm as follows:

Algorithm for building generalized Voronoi Diagram

- Using sweep-line algorithm. That is:
  - Sweep across S from above to below with a horizontal line L.
  - For site s_i != Γ, let x_i be the abscissa (x-coordinate) of $p(s_i)$ and define $E_L(i) = s_i \cap L$.
  - We maintain, for s_i whose point $p(s_i)$ has been sweept over, the site s_j where $E_L(j)$ is closest to x_i on L.
  - The unique disk with north pole $p(s_i)$ and touching (tangenting) s_j is computed, and minimal such disk for s_i so far, $D_L(i)$, is updated if necessary (if the new disk has smaller radiu).
  - The abscissa x_i is deactivated when $D_L(i)$ has been entirely sweept over by L. It is activated when L touching it.
  - As long as L intersects with $D_L(i)$, x_i is activated.

Algorithm for building generalized Voronoi Diagram

- Show some example on blackboard.

Algorithm for building generalized Voronoi Diagram

- Some observations:
  - If L sweeps over lowest point of $D_L(i)$, that means by this time point, we have not found another $D_L'(i)$ to replace (update) $D_L(i)$. That is the referred s_j of x_i has not been replaced by another $s_{j'}$ whose $E_L(j')$ is more closer to x_i than $E_L(j)$ of s_j.
  - If we update $D_L(i)$ to $D_L'(i)$, that means the radiu is smaller.
  - The $E_L(i)$ may consist of more than one component.

    (show on blackboard)

Algorithm for building generalized Voronoi Diagram

- Some observations:
  - Why we should deactivate x_1 when lowest point of $D_L(1)$ is sweept over?
    - Assume for site s_1, we have drawed $D_L(1)$ and L meets the lowest point of $D_L(1)$.
    - The center of $D_L(1)$ is always on vertical line through $p(s_1)$. Continue sweeping L, if we could have a D_L'(1) when x_1 is closer to $E_L(s_3)$ rather than $E_L(s_2)$ which is last closest E_L to x_1.
    - This disk should has larger radiu to D_L(s1). Since L is continuing sweeping and center of disk is on verticle line passing x_1.
    - Here, this new disk intersects with s_2.

Algorithm for building generalized Voronoi Diagram

- Lemma3. After completion of sweep, $D_L(i) = D_i$ holds for each s_i
- To keep small number of neighbor pairs (x_i, s_j) on L processed during the plane sweep, we only consider pairs where no other activated abscissa x_m lies between s_i and $E_L(j)$.
- The construction is of complexity O(nlogn).
  - We have n sites
  - For each sites we use logn to check/ update the STATUS tree.

Computing medial axis of augmented domain

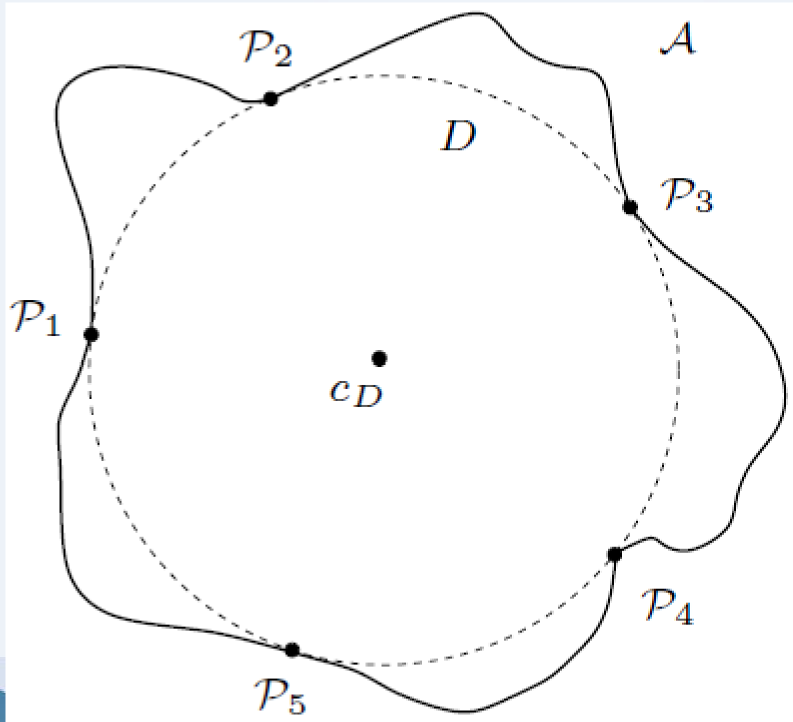
- Observation:
  - As has a connected boundray and we can travel from a point p on As, along As and back to p.
  - D_i is maximal inscribed disk which separate the domain into subdomains.
  - If we want to find MA (As), we can find MA(sub-As) and weave them together.
  - We use divide-and-conquer algorithm to compute MA(As).
  - The domain and its MA tree are split recursively until directly solvable base cases remain.

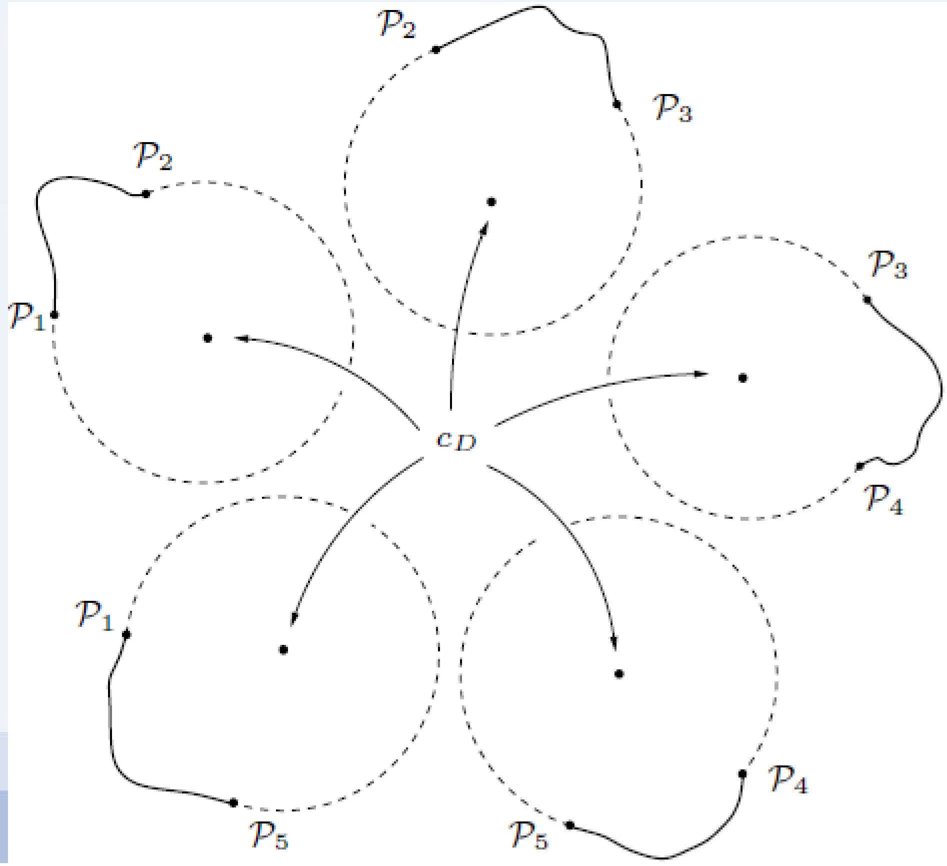
Computing medial axis of augmented domain

- Divide and conquer:
  - Divide step calculates a dividing disk and checks whether the induced decomposition is progressive. i.e. whether the resulting subdomains are combinatorially smaller (containing less arcs) the domain itself, Until we meet the base cases.
  - When we meet base cases, we use precomputed MA arcs as the output MA of base cases (base cases will be introduced soon).
  - Conquer step concatenates the already computed medial axis for the subdomains, as two ore more subdomain shares the same dividing circle certer.

Computing medial axis of augmented domain

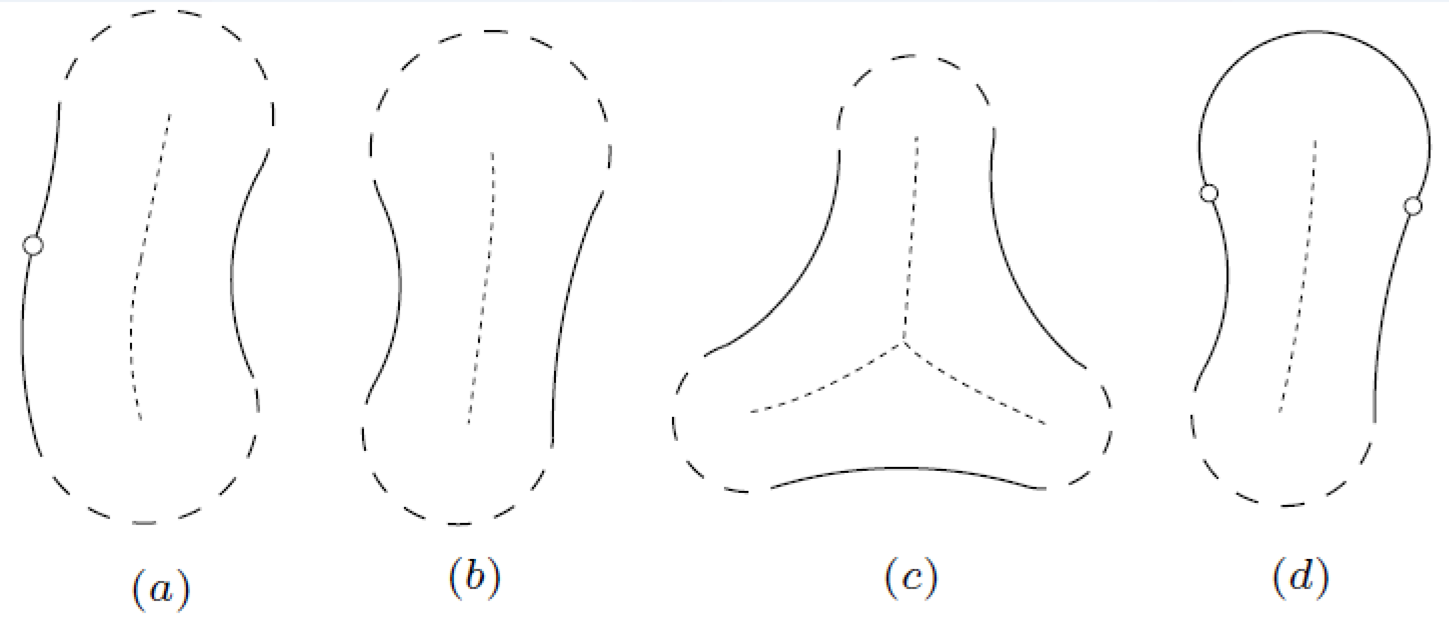
- Layers of dividing disk
  - Suppose we have a domain A, and it has a dividing disk C_D.
  - The number of layers of disk is the number of tangent point of C_D with A.





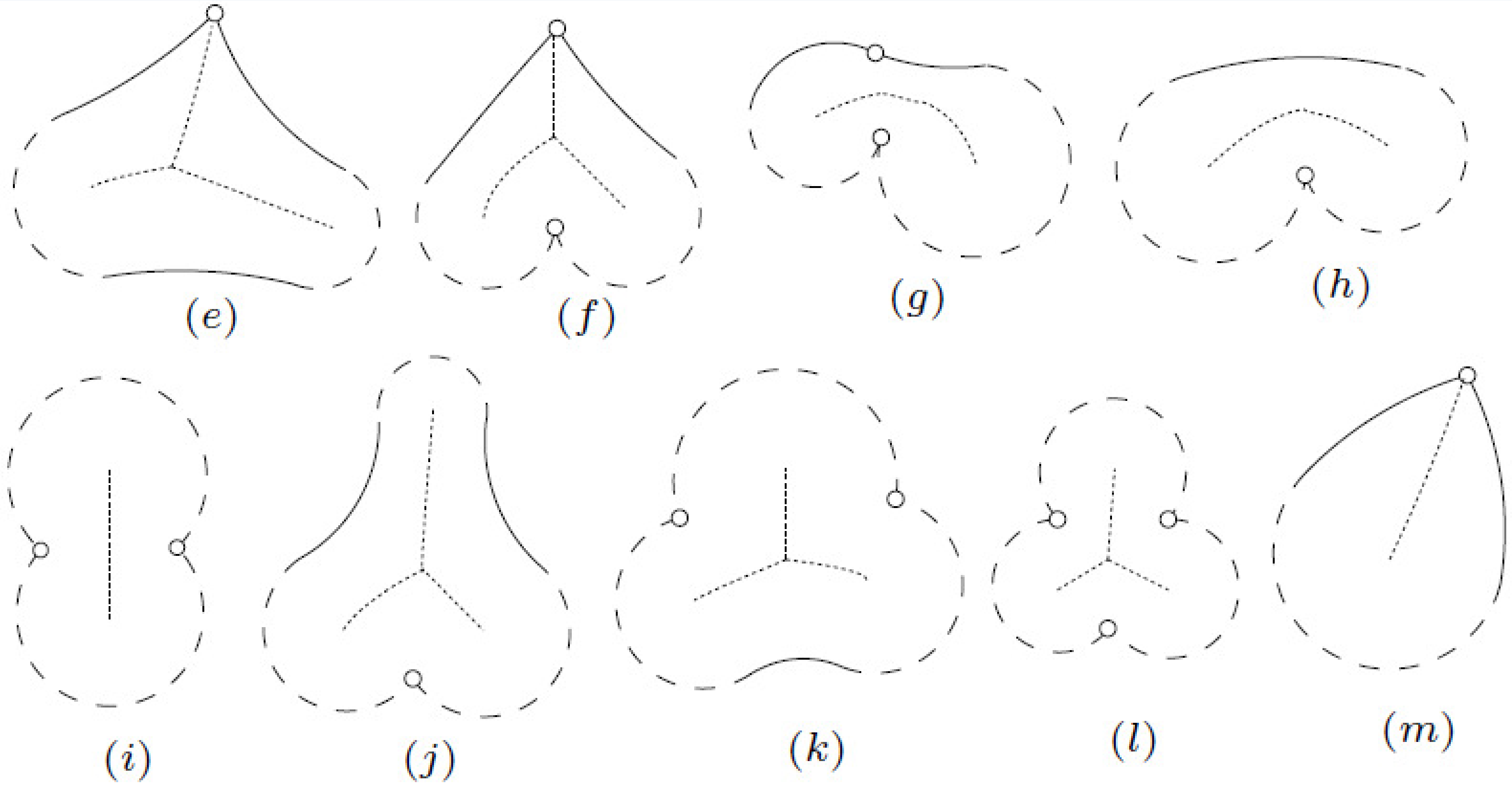
Computing medial axis of augmented domain

- Base cases：
  - For C^1 continunity boundray of sites, we have 4 types of base cases.



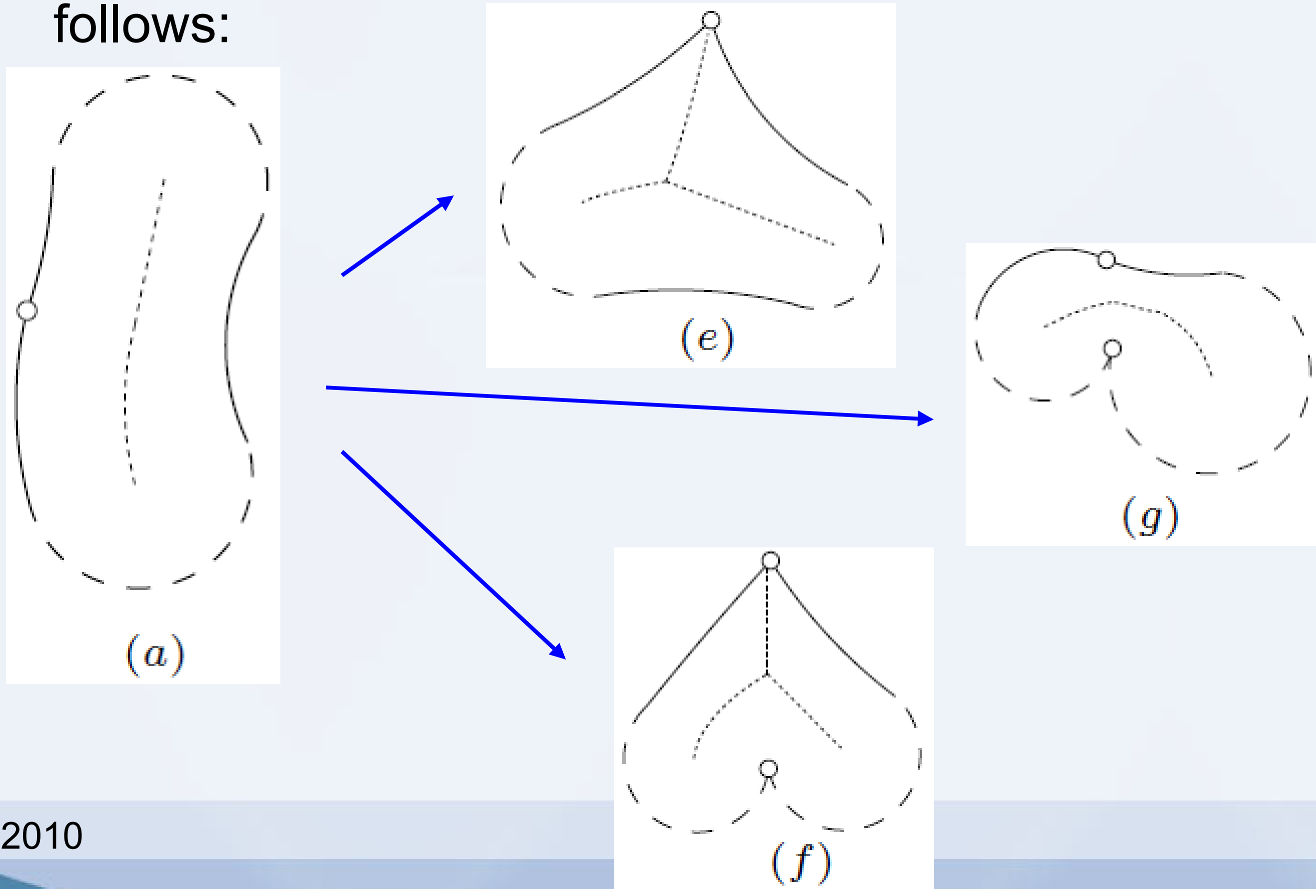
Computing medial axis of augmented domain

- Base cases：
  - For C^0 continunity boundray of sites, we have 9 types of base



Computing medial axis of augmented domain

- Base cases：
  - The relation of base cases between C^0 type and C^1 type is as follows:



Computing medial axis of augmented domain

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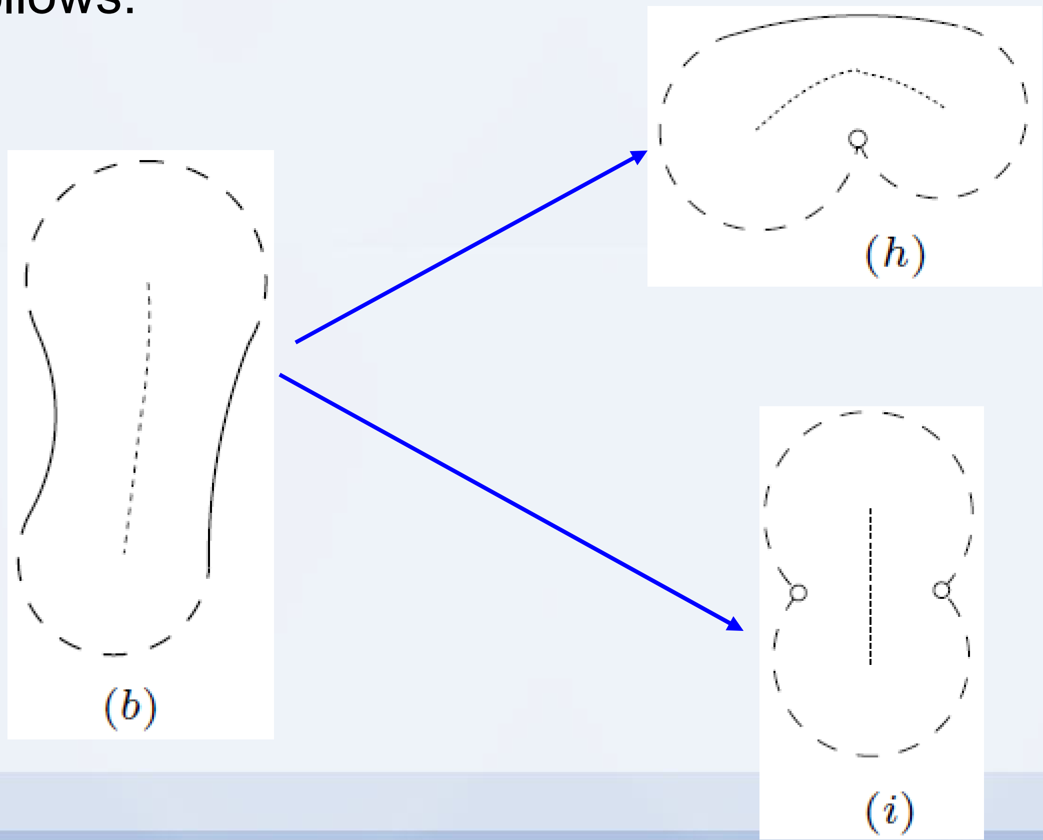

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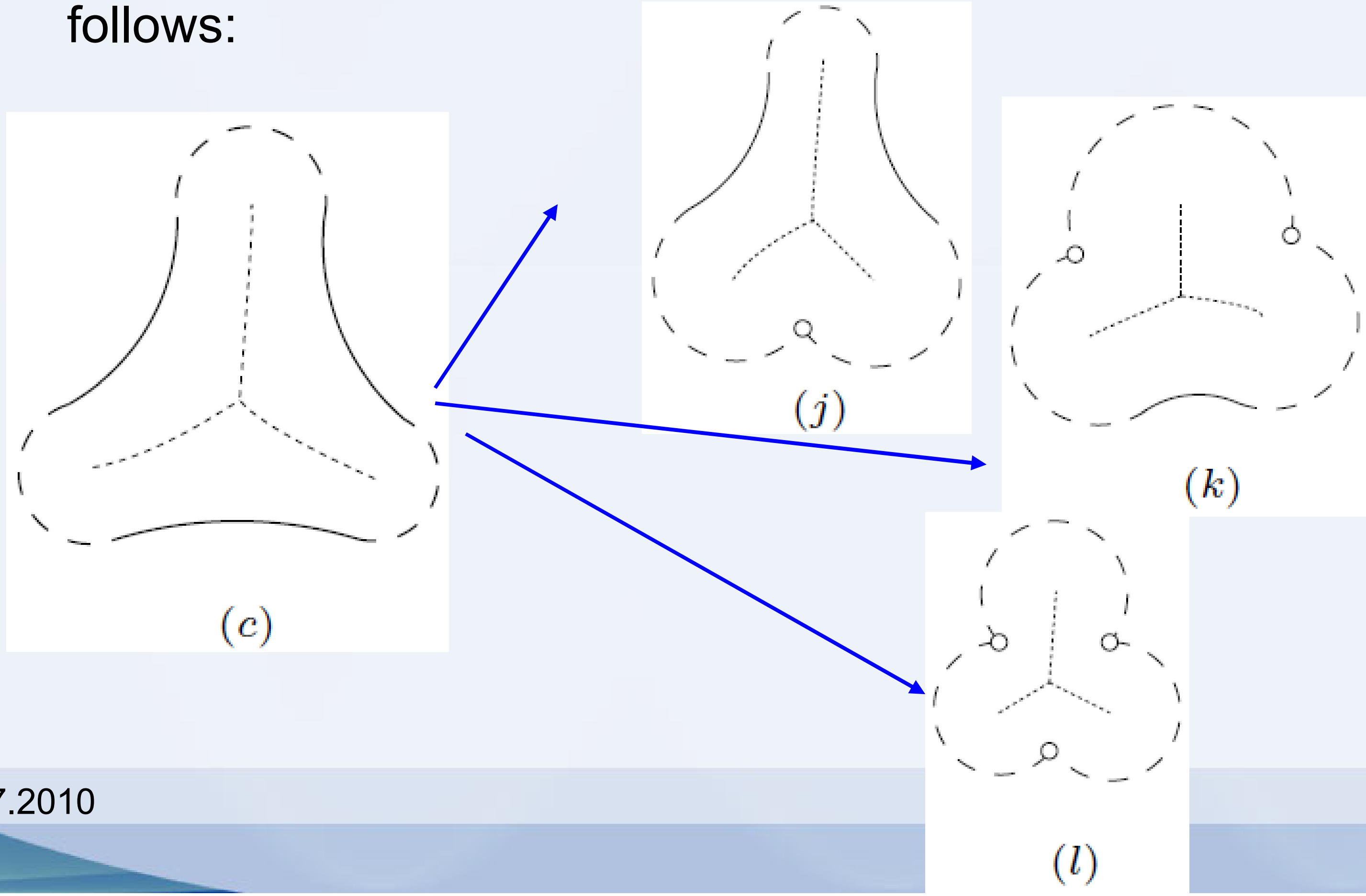

Computing medial axis of augmented domain

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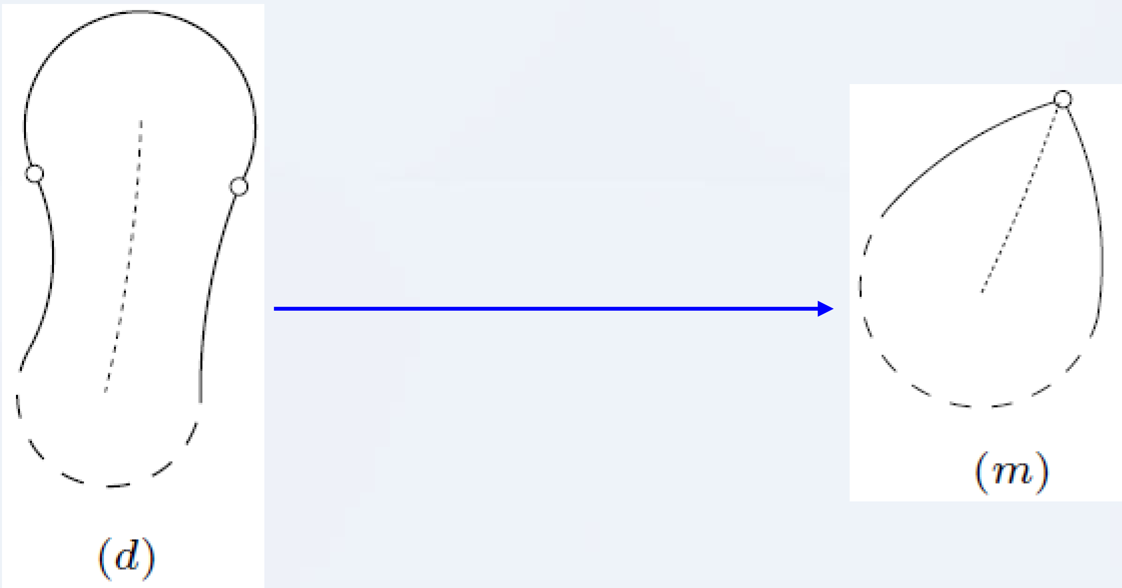

Computing medial axis of augmented domain

- How to compute split disk for As?
  - What we know?
    - ∂As is continuous and has direction. Traversal-enabled.
    - ∂As consists of bound inscribed disks (artificial arcs) and site boundrays (site segments)
  - What we do?
    - For point p on ∂ As, we compute maximal inscribed disk D for As and tangent with ∂ As at p.
    - Scan along ∂ As, when disk intersects with any other sites, its size shrinks. Do not change when intersects when artificial arcs.
    - Artificial arcs are only used to link site segment in correct cyclic order. Do not play any geometric role.

Computing medial axis of augmented domain

- How to compute split disk for As?

Algorithm 3 `maximaldisk`$(a, \partial\mathcal{A})$

{Compute a maximal disk on a in $\mathcal{A}$}

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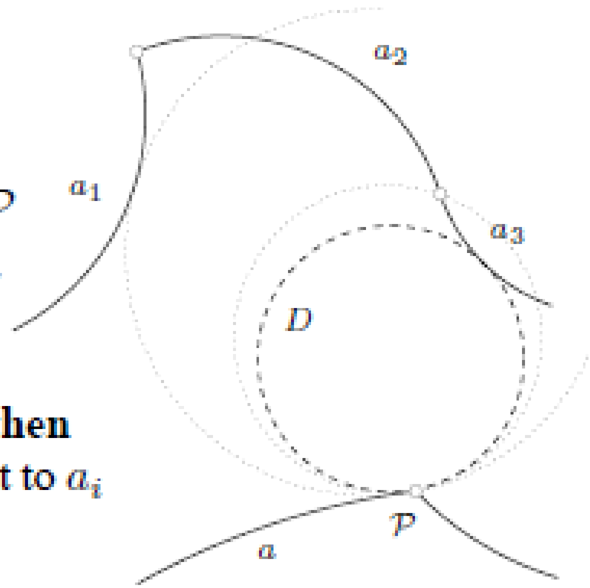
if a has a reflex endpoint **then**
    $\mathcal{P} \leftarrow$ reflex endpoint
else
    $\mathcal{P} \leftarrow$ midpoint of a
end if
$D \leftarrow$ halfplane tangent at $\mathcal{P}$
$k \leftarrow$ number of arcs on $\partial\mathcal{A}$
for $i = 1 \dots k$ **do**
    $a_i \leftarrow \mathrm{i}^{th}$ arc of $\partial\mathcal{A}$
    if $a \neq a_i \wedge D \cap a_i \neq \emptyset$ **then**
        $D \leftarrow$ disk at $\mathcal{P}$ tangent to a_i
    end if
end for
return D

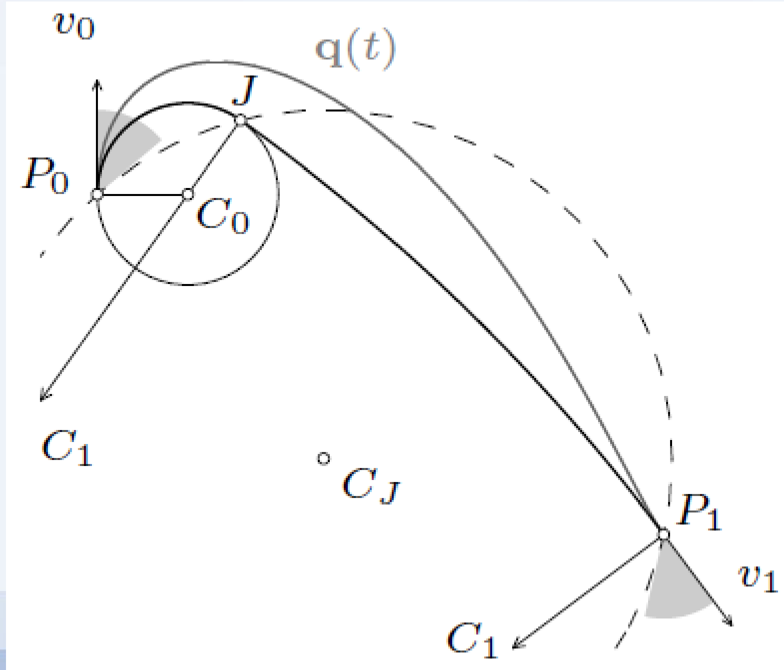


Computing medial axis of augmented domain

- Complexity
  - Assume we have n sites (n site segments).
  - If each site spend us O(1).
  - Then, computing disk takes O(n) time.

Experiments:

- Site appoximation:
  - We approximate the site boundarys as connected circular arcs.
- Biarc
  - It is the concatenation of two arcs which meet with a common tangent at a joint J.
  - Joint circle



Experiments:

- Biarc
  - Biarc types:
    - Equal chord (EC): arcs of equal length.
    - Parallel tangent (PT) : biarc makes the tangent at joint parallel to the line p_0p_1
    - Intersection (IS) : biarc determines J by intersecting the joint circle with the given curve.
    - Spiral (SP): biarc chooses one of the arcs as a segment of an osculating circle of the given curve
  - Why biarc?
    - It is for approximate boundray of sites
    - It could makes the # of self-edge decrease.
    - It is not hard to compute a disk that is tangent to two arcs. But may be hard to compute disk that is tangent to two arbitrary curves.
    - Comparing with polylines, biarc approximation largely decrease the amount of computation.

Experiments:

- Approximation quality of biarcs:
  - We measure it by **Hausdorff distance** which measures how far two subsets of a metric space are from each other.
  - Assume the biarc length (probably average) is h, Hausdorff distance decreases when decreasing h.
  - Taylor expansion of errors, k_i is the i-th derivative of the curve's curvature with respect to the arc length parameter at the point of intersest

Type	Maximal distance error (up to $O(h^5)$)
EC	$\max\left(\left\lvert\frac{\kappa_1}{324}h^3-\frac{\kappa_2}{1944}h^4\right\rvert,\left\lvert-\frac{\kappa_1}{324}h^3-\frac{7\kappa_2}{1944}h^4\right\rvert\right)$
PT	$\max\left(\left\lvert\pm\frac{\kappa_1}{324}h^3+\frac{6\kappa_1^2-\kappa_0\kappa_2}{1944\kappa_0}h^4\right\rvert\right)$
IS	$\max\left(\left\lvert\frac{\kappa_1}{324}h^3+\frac{7\kappa_2}{3888}h^4\right\rvert,\left\lvert-\frac{\kappa_1}{324}h^3-\frac{5\kappa_2}{3888}h^4\right\rvert\right)$
SP	$\left\lvert-\frac{\kappa_1}{96}h^3-\frac{\kappa_2}{192}h^4\right\rvert$

Experiments:

- Five complex sites bounded by n circular arcs (average value over 40 different inputs)
- Ratio n $\log_2 n$ = atomic-steps/ ($n\log_2 n$)
- Ratio n $(\log_2 n)^2$ = atomic-steps/ ($n (\log_2 n)^2$)

n	atomic steps	ratio $n \log_2 n$	ratio $n(\log_2 n)^2$
507	6620	1.45	0.16
2070	32892	1.44	0.13
5196	91649	1.43	0.12
10474	199001	1.42	0.11
20488	417839	1.42	0.10
172198	4223178	1.41	0.09

Experiments:

- Uniform distribution of n point sites

n	atomic steps	ratio $n \log_2 n$	ratio $n(\log_2 n)^2$
400	7591	2.20	0.25
2000	54662	2.49	0.23
4000	143391	3.00	0.25
20000	1015149	3.55	0.25
40000	2659149	4.35	0.28
200000	19820012	5.63	0.32

Experiments:

- Compare with CGAL
  - Site is polygon and bounded by totally n line segments.

n	ALG	CGAL
1714	0.3	6.3
5622	1.1	42.6
25210	4.7	651.6
116460	23.5	19650.5
250366	57.8	$>$ 24 h
537360	131.1	$>$ 1 week

(a) 40 complex sites

n	ALG	CGAL
100	0.14	0.26
500	0.8	1.5
1000	2.2	3.1
5000	13.7	18.5
10000	39.3	37.6
50000	395.3	201.6

(b) Many small sites

Applications

- Motion planning
- Trimmed offsets

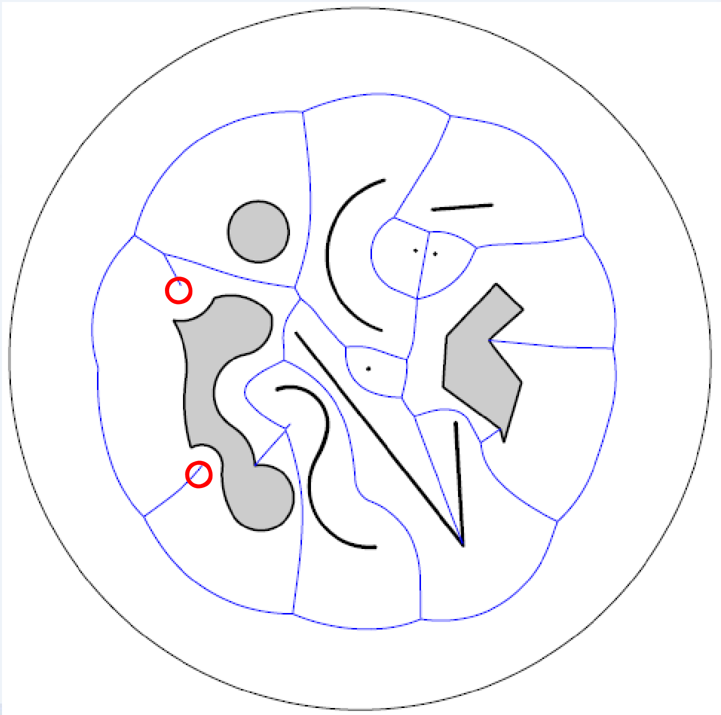
Motion planning

- Bound circle (box) is the bound for region.
- Sites are the obstacles
- Want to find path from source to target points.

- Derive V(S) as path network
- For each base path section (path between two adjacent voronoi verticies), maintain minimal diamiter of inscribed disk. And measure it with traveler's size when traveler start to consider to travel along this sub path.
- If source/ target is on MA, find path with measurement, if not, they should be in some region.
- Just find point on region bound s' / t' and find path s' to t' with measurement.
- Finally, if found, connect s with s' and t with t'.

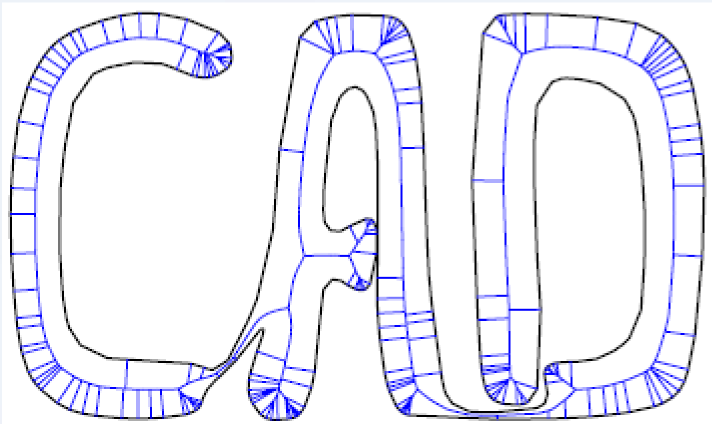
Motion planning

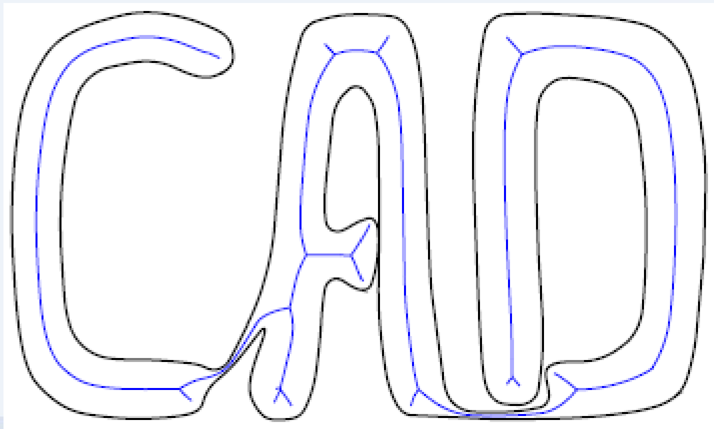
- Self-edge
  - Edge of Gs, points of it have equal distance to more than one points on same site.
  - It leads to blind lane.



Motion planning

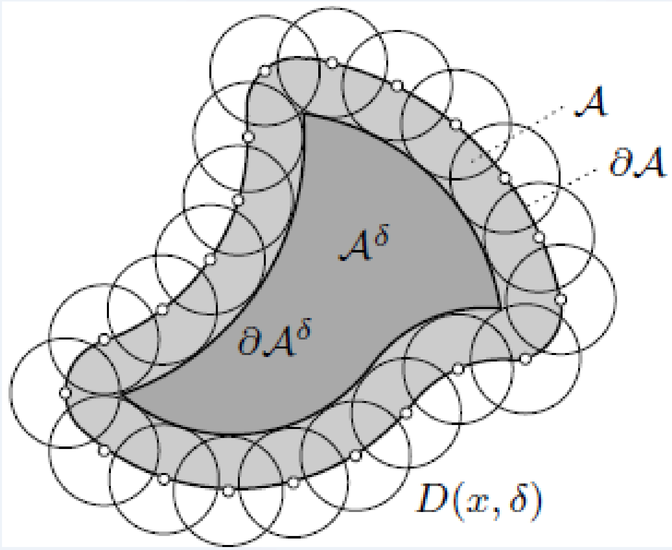
- Self-edge
  - Site approximation decreases the number of self-adges.





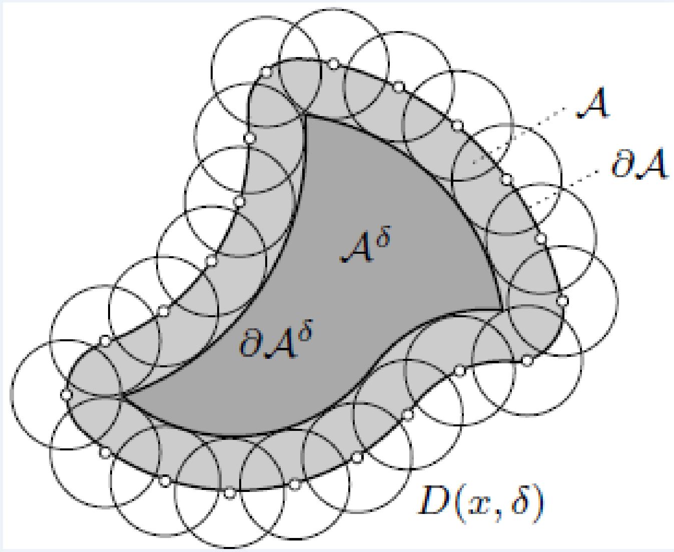
Trimmed offsets

- Compute inner or outer offsets of a planar shape A.
- For inner offsets, take outer boundray of A as the surrounding curve ∂A the holes of A as sites.
- For outer offsets, compute inner offsets of complement of A.



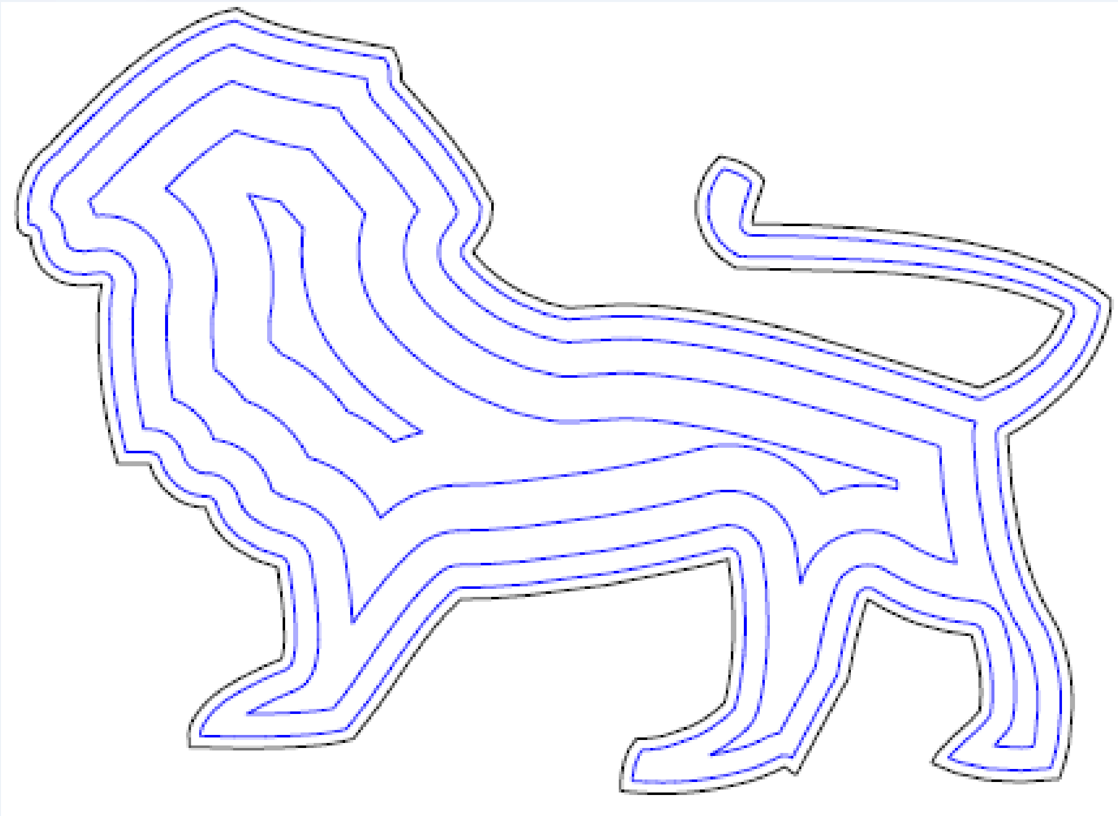
Trimmed offsets

- Offsets at distance δ is: $$\mathcal{A}^{\delta} = \mathcal{A} \setminus \bigcup_{x \in \partial A} D(x, \delta)$$
- D(x, δ)is the disk with center x and radiu δ



Trimmed offsets

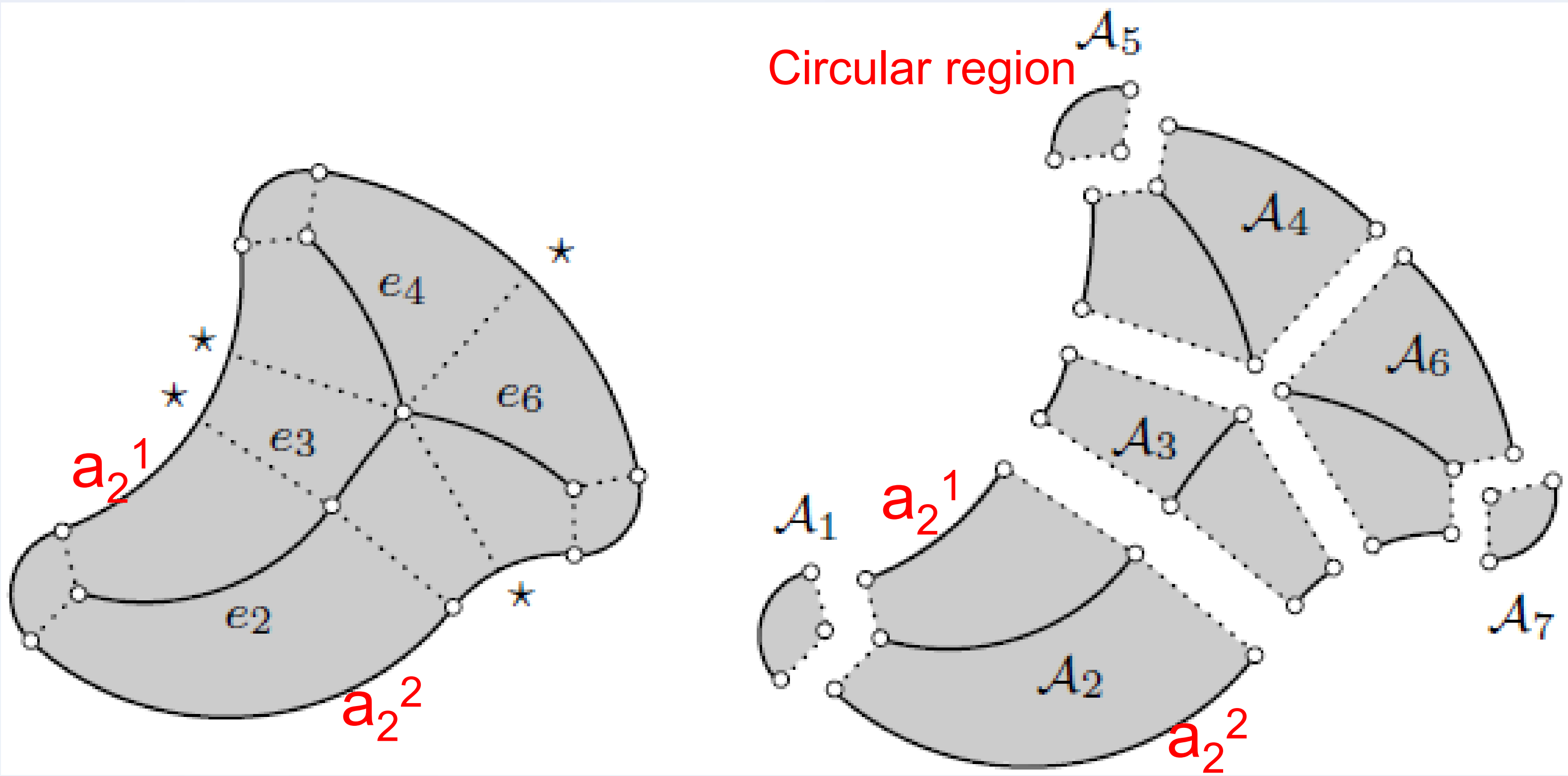
- Inner offsets for different values of δ



Trimmed offsets

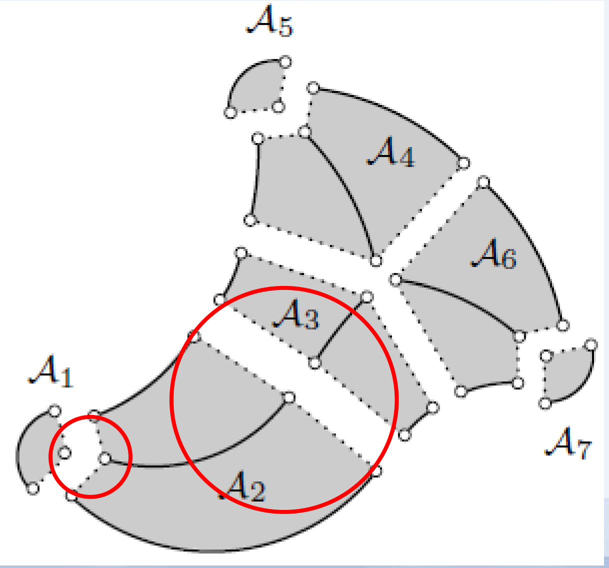
- Subshape of A. edge graph Gs consists of conic segments e_i, each being the bisector of two arcs a_i^1 and a_i^2.
- For a point x on either arc, consider the segment which is in A and connects x with e_i.
- Union of these line segments forms the subshape $A_i \subseteq A$ associated with e_i.
- Each circular region consisting of all line segments which connect the points of the arc with its center.

Trimmed offsets



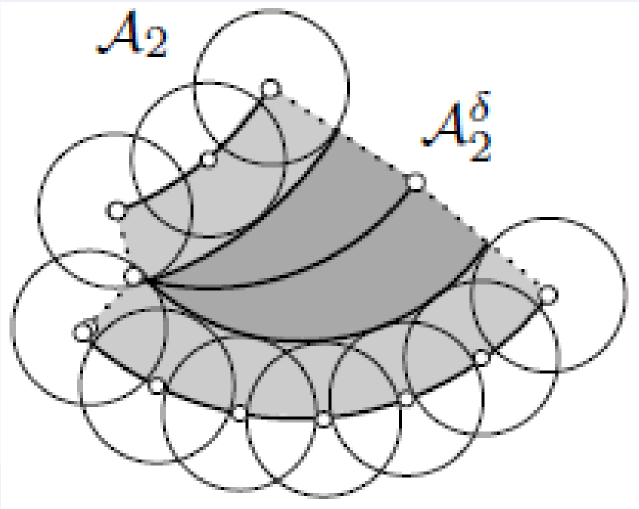
Trimmed offsets

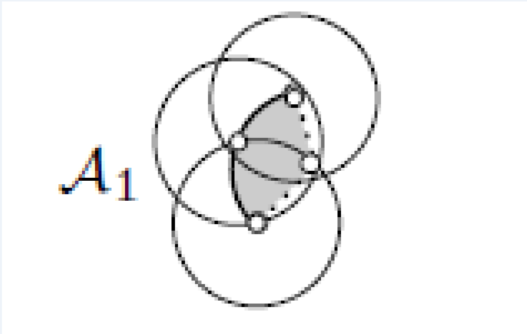
- Monotonic of subshape A_i:
  - The radii of the maximal disks of A with centers on e_i have no inner extrema.
  - Inner extrema r_{min}, r_{max} shown below:
    - r_{min} : radiu of smallest disk centered on e_i and tangent to a_i^1, a_i^2
    - r_{max} : radiu of largest disk centered on e_i and tangent to a_i^1, a_i^2.



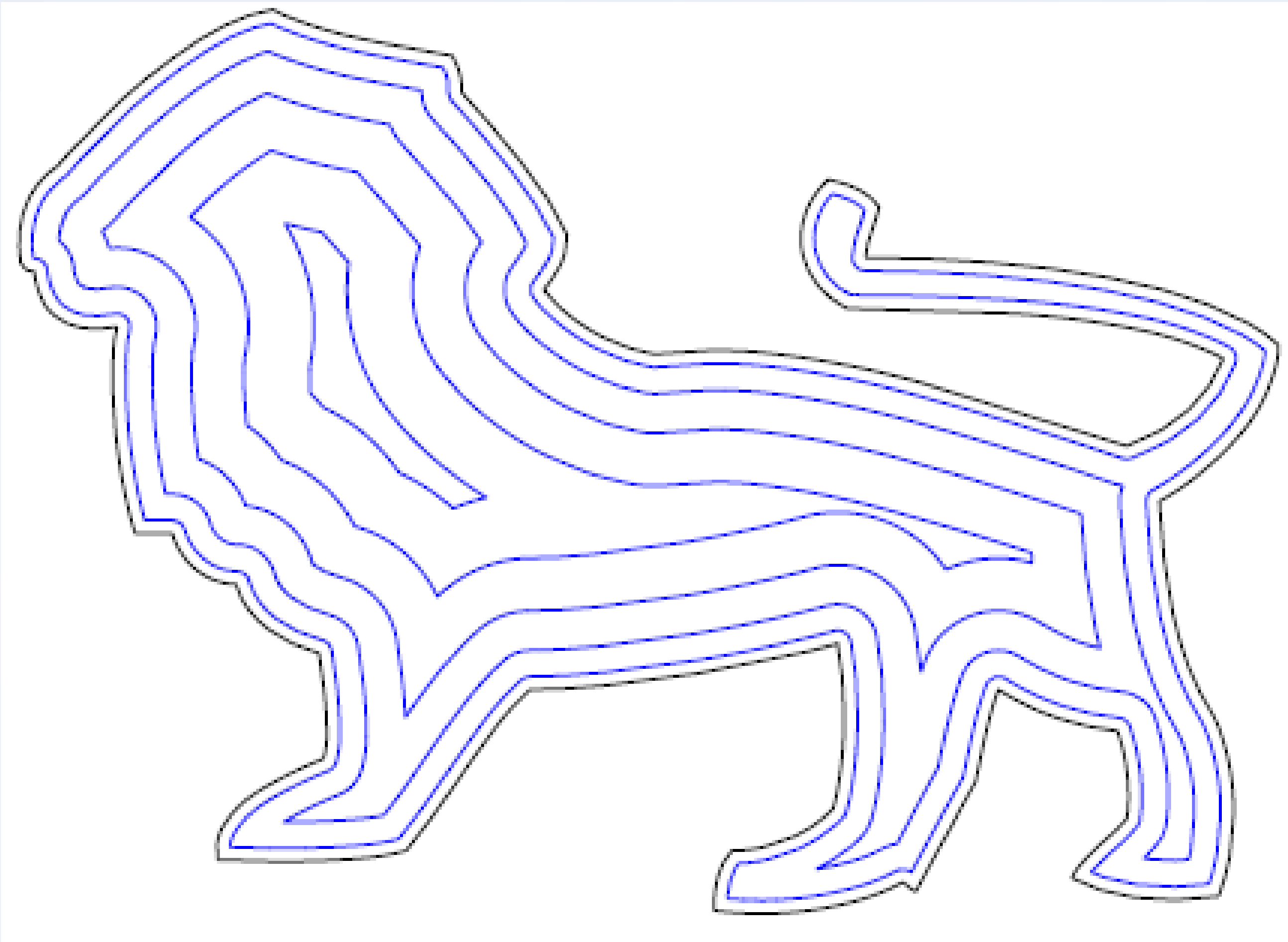
Trimmed offsets

- Offsets is done separetely for each monotonic subshapes
  - $\delta < r_{min}$, then the offsets of arcs at distance δ are fully contained in ∂A^{δ}
  - $r_{min} < \delta < r_{max}$, the offset arcs are trimmed at their intersections.
  - $r_{max} < \delta$, this subshape will be entirely trimmed off.





Trimmed offsets



Thank you!

Questions?

Comments?