Homework 11

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This assignment is **due on July 8/9** in your tutorial session. You are allowed (even encouraged) to discuss these problems with your fellow classmates. All submitted work, however, must be *written individually* without consulting someone else's solutions or any other source like the web.

Problem 1 **(4 points)**

Consider the integer programming problem:

$$\begin{aligned} \min\ & x_{n+1} \\ \text{s.t. }\ & 2x_1 + 2x_2 + \ldots + 2x_n + x_{n+1} = n \\ & x_i \in \{0,1\}, \qquad i = 1, \ldots, n+1 \end{aligned}$$

(a) What are possible objective values of integer feasible solutions when n is odd?

(b) What is the optimal value of the LP relaxation in which no more than $n/2$ of the variables $x_1, \ldots, x_n$ are fixed (by a branch and bound algorithm) to 0 or 1, when n is odd? Can the corresponding optimal solution be integer?

(c) Using (a) and (b), argue that any branch and bound algorithm that uses the LP relaxation to compute lower bounds, and branches by setting a fractional variable to either 0 or 1, will require the enumeration of an exponential number of subproblems when n is odd.

Problem 2 **(1+2+2 points)**

Formulate the following problems as integer linear programs. How many constraints and variables did you use?

(a) The independent set problem: Given an undirected graph $G = (V, E)$, find a maximum independent set, that is, a maximum subset of vertices in which no two vertices are adjacent.

(b) The facility location problem: We are given n potential facility locations and a list of m clients who need to be serviced from these locations. There is a fixed cost f_j of opening a facility at location j, while there is a cost c_{ij} of serving client i from facility j. The goal is to select a set of facility locations and assign each client to one facility, while minimizing the total cost.

(c) The Sudoku problem: Given is an $n^2 \times n^2$ matrix, $n \in \mathbb{Z}^+$, which contains some given integral entries between 1 and n^2. The task is to fill the remaining entries with integers from $\{1, 2, \ldots, n^2\}$, such that each row, each column, and each of the n^2 major $n \times n$ submatrices contains each integer 1 through n^2 exactly once.

Problem 3 **(4 points)**

Show that the following simple combination of the *trivial algorithm* and the *randomized rounding algorithm* for Max-SAT presented in the lecture yields a 4/3-approximation: flip a fair coin and run the trivial algorithm in case of heads and the randomized rounding algorithm in case of tails.

The two crucial lemmata from the lecture are repeated in the following; for notation see the lecture notes.

Lemma 1 *The trivial algorithm yields for any clause $C_i, i = 1, \ldots, m$,*

$$\mathbb{E}\,[\,W_i\,] \;=\; \left(1 - \frac{1}{2^{k(i)}}\right) w_i.$$

Lemma 2 *The randomized rounding algorithm yields for any clause $C_i, i = 1, \ldots, m$,*

$$\mathbb{E}\,[\,W_i\,] \;\geq\; \left(1 - \left(1 - \frac{1}{k(i)}\right)^{k(i)}\right) w_i\, z_i^{LP}.$$