Total Dual Integrality

Def. A \vec{x} \leq \vec{b} is called totally dual integral (TDI) if for each \vec{c} \in \mathbb{Z}^n with \max \{ \vec{c}^T \vec{x} \mid A \vec{x} \leq \vec{b} \} < \infty holds:
\[
\max \{ \vec{c}^T \vec{x} \mid A \vec{x} \leq \vec{b} \} = \min \{ \vec{y}^T \vec{b} \mid \vec{y} \vec{A} = \vec{c}^T, \vec{y} \geq 0, \vec{y} \in \mathbb{Z}^m \}.
\]

This gives primal integrality "for free".

Cor. Let \vec{A} \in \mathbb{Q}^{m \times n}, \vec{b} \in \mathbb{Z}^m, and \vec{A} \vec{x} \leq \vec{b} TDI \Rightarrow \exists \vec{x} \mid \vec{A} \vec{x} \leq \vec{b} \text{ integral.}

Pf. Def + Thm.
(\vec{b} \in \mathbb{Z}^m \Rightarrow \vec{y}^T \vec{b} \text{ with } \vec{y} \in \mathbb{Z}^m \text{ is integral again.)

Application/Example

Given: directed graph \( D = (V, A) \), s.t. \( \vec{v} \in V \)
\( P = \{ \text{ s-t paths } x_a \in D \} \), \( C: A \rightarrow \mathbb{R}_+ \)

Task: Assign weights \( y_a \geq 0, a \in A \), s.t. weight on any s-t path is at least 1.

Obj.: \( \min \sum_{a \in A} c(a) y_a \)

LP formulation:
\[
\begin{align*}
\min \sum_{a \in A} & c(a) y_a \\
\text{s.t. } & \sum_{a \in P} y_a \geq 1, P \in P \\
& y_a \geq 0, a \in A
\end{align*}
\]

Claim: \( Q \) is integral because it is TDI.

dual:
\[
\begin{align*}
\max \sum_{P \in P} & 1 \cdot x_P \\
\text{s.t. } & \sum_{P \in P} x_P \leq c(a), A \in A \\
x_P \geq 0, A \in P
\end{align*}
\]

This is a formulation of the max s-t flow problem. If the "capacities" \( c(a), a \in A \), are integral,
then there exists an integral max flow.

\[]

Coroll. above implies that \( Q \) is integral.
Remark: TDIness is not a property of polyhedra but a property of linear systems. There can be two different descriptions of a particular polyhedron by inequ. systems s.t. one is TDI and the other one is not.

Ex. \[
\begin{pmatrix}
1 & 1 \\
1 & 0 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\ x_2
\end{pmatrix}
\leq
\begin{pmatrix}
0 \\ 0
\end{pmatrix}
\] [TDI]

\[
\begin{pmatrix}
1 & 1 \\
1 & -1 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\ x_2
\end{pmatrix}
\leq
\begin{pmatrix}
0 \\ 0
\end{pmatrix}
\] [not TDI]

Generally TDI systems contain more constraints than necessary for just defining the polyhedron.

Adding valid ineq. does not destroy TDI ness.

Prop. If \( Ax \leq b \) is TDI and \( a^T x \leq \beta \) is valid ineq. for \( \exists x : Ax = b \)
then \( Ax \leq b, a^T x \leq \beta \) is also TDI.

Pf. For \( C \in \mathbb{Z}^n \):

\[
(*) = \min \left\{ y^T b \mid y^T A = c^T, y \geq 0 \right\} = \max \left\{ c^T x \mid Ax \leq b \right\}
\]

\[
= \max \left\{ c^T x \mid Ax \leq b, a^T x \leq \beta \right\} \tag{duh}
\]

\[
= \min \left\{ y^T b + y^T \beta \mid y^T A + \delta^T a = c^T, y, \delta \geq 0 \right\}
\]

\[
= (**)
\]

(*) has opt. integral solution \( y^* \) hence \( Ax \leq b \) is TDI.

\( \Rightarrow \) (***) has opt. integral solution \((y^*, 0) = (y, \delta^*)\).

\( \Box \)

Cor. \( Ax \leq b, x \geq 0 \) is TDI if \( \min \left\{ y^T b \mid y^T A \geq c, y \geq 0 \right\} \) has integral opt. solution \( y \) for each \( C \in \mathbb{Z}^n \) for which \( \min \) is finite.

\( Ax = b, x \geq 0 \) is TDI if \( \min \left\{ y^T b \mid y^T A \geq c \right\} \) has integral opt. solution \( y \) for each \( C \in \mathbb{Z}^n \) for which \( \min \) is finite.
Existence of TDI description with integral $A$?

**Theorem:** For each rational polyhedron $P$, there exists a rational TDI-system $Ax \leq b$ with $A$ integral and $P = \{ x \mid Ax \leq b \}$. In particular, $b \in \mathbb{Z}^m$ (so $P$ integral).

**Connection between TDI system and integral polyhedra.**

1. "Procedure" to prove integrality of polyhedra (justified by prev. Thm.)
   - Find appropriate defining system $Ax \leq b$ with $A, b$ integral.
   - Prove $Ax \leq b$ TDI.
   - From Thm. follows that $\{ x \mid Ax \leq b \}$ integral.

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**Question:** Are there matrices $A$ s.t. $Ax \leq b, x \geq 0$ is TDI for each $b \in \mathbb{Z}^m$?

**Totally unimodular matrices**