Totally unimodular matrices

Def. A matrix $A$ is totally unimodular if each subdeterminant of $A$ is $0$, $1$, or $-1$. Clearly, $A \in \mathbb{R}^{n \times n}$.

(A subdeterminant of $A \in \mathbb{Z}^{m \times n}$ is $\det B$ for some square submatrix $B$ of $A$ obtained by choosing an appropriate number of rows and columns of $A$.)

Thus (Hoffman, Konvaling '56)

$A \in \mathbb{Z}^{m \times n}$ totally unimodular $\iff \exists \, P = \{ x | Ax \leq b, x \geq 0 \}$ integral for any $b \in \mathbb{Z}^m$.

Pf. 

"\Rightarrow" Let $A \in \mathbb{Z}^{n \times n}$, $b \in \mathbb{Z}^n$, and $x$ a vector in $P$.

$x$ is solution of some subsystem $A'x = b'$ for some subsystem $A'x \leq b'$ of $(A^\top x) \leq (b^\top)$, with $A' \in \mathbb{Z}^{k \times k}$ non-singular, $(\Rightarrow \det A' \neq 0)$

$A$ totally \iff $\det A' = 1$ \iff $x$ is integral.

Cramer's rule: $x_i = \frac{\det (A'_{i,:})}{\det A'}$ integral for integral $b \rightarrow A'_{i,:}$.

"\Leftarrow" Suppose all vertices of $P$ are integral $\forall b \in \mathbb{Z}^m$.

Let $A' \in \mathbb{Z}^{k \times k}$ be non-sq. submatrix of $A$.

To show $|\det A'| = 1$. W.l.o.g.

$A = \begin{pmatrix} A' & \ast \\ \ast & \ast \end{pmatrix}$

$(A \cdot I_m) = \begin{pmatrix} A' & I_k & 0 \\ \ast & \ast & 0 \\ \ast & \ast & I_{m-k} \end{pmatrix}$

$B \in \mathbb{Z}^{m \times m}$

$\det A' = \det B$. 

To show \( |\text{det} A^1| = |\text{det} B| = 1 \), prove that \( B^{-1} \) is integral.

Why? \( B \cdot B^{-1} = I = \text{det} B \cdot \text{det}(B^{-1}) = 1 \)

\( B \) is integral \( \iff \text{det} B \) is integral \( \iff \text{det}(B^{-1}) = 1 \)

Let \( n \in \mathbb{Z}_+ \) and show that \( B^{-1} e_i \in \mathbb{Z}^m \).

Choose \( y \in \mathbb{Z}^m \) s.t. \( z = y + B^{-1} e_i \geq 0 \).

Then \( b = Bz = By + e_i \) is integral.

Add rows to \( z \rightarrow z' \) with \((A_{lm}) z' = Bz = b \in \mathbb{Z}^m\).

\[
(A_{lm}) z' = \begin{pmatrix} A^1 \ast & I_m \ast & 0 \\ * \ast & 0 \ast \end{pmatrix} \begin{pmatrix} z_1 \\ 0 \\ \vdots \\ z_m \end{pmatrix} = b.
\]

Let \( z'' \) consist of first \( n \) entries of \( z' \).

\[
\begin{pmatrix} A^1 \ast \\ * \ast \end{pmatrix} z'' = \begin{pmatrix} b \ast \\ 0 \ast \end{pmatrix} \quad \text{with} \quad z'' \geq \begin{pmatrix} 0 \ast \\ 0 \ast \end{pmatrix}.
\]

Furthermore, \( z'' \) satisfies \( T \) with equality \( \iff \) the first \( k \) rows and the last \( n-k \) rows are \( \text{lin. indep.} \).

\( z'' \) is a vector of \( \mathbb{Z}^m \) \( \iff \) \( z'' \in \mathbb{Z}^m \) \( \iff \) \( z' \in \mathbb{Z}^{m+n} \) \( \iff \) \( z \) is integral.

\[
\square
\]
Then. Let $A \in \mathbb{Z}^{m \times n}$. The following statements are equivalent:

(i) $A$ is totally unimodular.

(ii) $\forall b \in \mathbb{Z}^m$, $\forall c \in \mathbb{Z}^n$:

$$\max \{ c^T x \mid Ax \leq b, x \geq 0 \} = \min \{ y^T b \mid y^T A \geq c^T, y \geq 0 \}$$

have integral solutions $x$ and $y$ (if finite).

(iii) $Ax \leq b, x \geq 0$ is TDI for all $b \in \mathbb{R}^m$.

(iv) $\forall R \subseteq \{1, -1\}^n$ 1-partition $R = R_1 \cup R_2$ (disjoint):

$$\sum_{i \in R_1} a_{ij} - \sum_{i \in R_2} a_{ij} \in \{ -1, 0, 1 \} \quad \forall j \in \{1, -1\}^n$$

Corollary: The node-edge incidence matrix of an undirected graph is totally unimodular if and only if the graph is bipartite. 

This follows directly from (iv) $\Rightarrow$ (i).

\[ \begin{vmatrix} e_1 & e_2 & \cdots & e_n \\ v_1 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix} \]

\[ R_1 \quad \text{resp. any subset of nodes} \]

Then. Let $A \in \{0, 1, -1\}^{m \times n}$, where each column has at most one +1 and at most one -1. Then $A$ is TDI.

Proof. Let $N$ be the submatrix of $A$. Induction on $k$:

$k = 1$: $\det N \in \{0, 1, -1\}$ \( \checkmark \)

$k \geq 2$:

(i) $N$ has at least one column with at most one non-zero entry.

\[ = \det N = \pm 1, \det N' \text{ for } (k-1) \times (k-1) \text{ matrix} \]

\[ = \det N \in \{0, 1, -1\} \text{ by induction hypothesis} \]

(ii) $N$ has all columns with more than one non-zero entry (one +1, one -1).

Then sum of all rows gives $(0, \ldots, 0)$.

\[ \Rightarrow \text{lin. dependent} = \det N = 0. \]
**Corollary:** The node-edge incidence matrix of any digraph is Trivial.

It follows directly from prev. Thus,

\[ D = (V, A) \quad \neq \quad \begin{pmatrix}
  v_1 & a_1, a_2, \ldots, a_n \\
  v_2 & \begin{pmatrix} +1 & 0 \\
  -1 & 0 \\
  0 & 0 \\
  0 & 0
  \end{pmatrix} \\
  \vdots & \ddots \\
  v_n & \end{pmatrix} \]

\[ v_1 \xrightarrow{a_i} v_2 \]

**Theorem:** A matrix \( A \in \{0, 1\}^{m \times n} \) has the consecutive ones property (along columns), if in every column the 1's appear consecutively (assuming some linear ordering of rows of \( A \)). Any matrix with the consecutive ones property is Trivial.

**Proof:** [Homework]