Algorithmic Game Theory
Exercise Sheet 9

Exercise 1 (Theorem 15.8) *For every $\gamma \leq 1$, a cost sharing game $(\mathscr{A}, c)$ with transferable utilities has a non-empty γ-core if and only if for every balanced collection of weights λ, we have $\sum_{S \subseteq \mathscr{A}} \lambda_S c(S) \geq \gamma c(\mathscr{A})$.*

Exercise 2 (Integrality Gap) *Let $G = (V, E)$ be an undirected graph and let $w : E \to \mathbb{R}_{\geq 0}$ be a non-negative weight function on the edges. A matching M is a set of edges no two of which share an endpoint; the weight of a matching is the sum of the weights of its edges. Let x_e be an indicator variable for edge e. The following ILP captures the matching problem:*

$$
\begin{array}{lll}
\textit{maximize} & \displaystyle\sum_e w_e x_e & \\
\textit{subject to} & \displaystyle\sum_{e \in \delta(v)} \leq 1 & \textit{for all } v \in V \\
 & x_e \in \{0, 1\} & \textit{for all } e \in E;
\end{array}
$$

here $\delta(v)$ denotes the set of edges incident on v. The corresponding LP is obtained by replacing the constraint $x_e \in \{0, 1\}$ by $x_e \geq 0$. Let $\mathrm{OPT}_{\mathrm{ILP}}$ be the objective value of ILP and let $\mathrm{OPT}_{\mathrm{LP}}$ be the objective value of the corresponding LP. The purpose of this exercise is to show that the integrality gap of this LP is at most 3/2, i.e.,

$$\mathrm{OPT}_{\mathrm{LP}} \leq 3/2 \cdot \mathrm{OPT}_{\mathrm{ILP}}.$$

1. *Give an example where $\mathrm{OPT}_{\mathrm{LP}}$ is larger than $\mathrm{OPT}_{\mathrm{ILP}}$.*
2. *Let $(x_e)_{e \in E}$ be an optimal solution of LP. Consider the graph $G' = (V', E')$ where E' consists of all edges $e \in E$ with $0 < x_e < 1$.*
3. *Assume first that G' contains no cycle of even length.*
   - *Show that G' must consist of disjoint cycles of odd length.*
   - *Conclude that $x_e = 1/2$ for all $e \in E'$.*
   - *Conclude $\mathrm{OPT}_{\mathrm{LP}} \leq 3/2 \cdot \mathrm{OPT}_{\mathrm{ILP}}$.*
4. *Assume next that G' contains a cycle C of even length.*
   - *Show how to modify the solution $(x_e)_{e \in E}$ such that the objective value does not decrease and the number of edges in E' decreases by at least one.*

Exercise 3 (submodular is equivalent to decreasing marginal cost) *A function c mapping subsets S of $\mathscr{A}$ to real numbers is* submodular *if for all sets S and T*

$$c(S) + c(T) \geq c(S \cup T) + c(S \cap T). \tag{1}$$

A function c has the decreasing marginal cost property *if for all $R \subseteq \mathscr{A}$ and all $i, j \in \mathscr{A} \setminus R$ and $i \neq j$*

$$c(R \cup \{i\}) - c(R) \geq c(R \cup \{i, j\}) - c(R \cup \{j\}).$$

We assume $c(\emptyset) = 0$.

Exercise 4 (cut-values are submodular) *Let $G = (V, E)$ be an undirected graph. For a subset S of the vertices, let $c(S)$ be the number of edges in G having exactly one endpoint in S. Show that the function c is submodular.*

Exercise 5 (connection costs in rooted trees are submodular) *Let $T = (V, E)$ be a tree with root r and let w be a non-negative weight function on the edges. For a subset S of the vertices, let $c(S)$ be the cost of the smallest tree connecting the vertices in $S \cup \{r\}$. This tree is the union of the paths from r to v for $v \in S$.*

- *Show that c is submodular.*
- *Let $S \subseteq V$. Characterize the cost shares $\xi(i, S)$, $i \in S$ computed by algorithm Submodular-CostShares?*