Algorithmic Game Theory
Lecture on July 18th

Kurt Mehlhorn and Rob van Stee

July 14, 2011

We discuss the paper *Online Primal-Dual Algorithms for Maximizing Ad-Auctions Revenue* by Buchbinder, Jain, and Naor [BJN07]. The paper contains a wealth of results. The many results are a refinement of a basic result which we are going to treat.

The scenario is as follows. We have an on-line auction with n bidders and m items; i ranges over bidders and j ranges over items. Each bidder i has a non-negative budget B_i. When an item arrives, the bidders provide non-negative bids b_{ij}. The mechanism assigns the item to one or none of the bidders. The mechanism has to ensure that the bidders stay within their budget limits, i.e., if M_i is the set of items assigned to i then $\sum_{j \in M_i} b_{ij} \le B_i$. The goal of the mechanism is to maximize the total revenue. The off-line version of the problem is captured in the following integer linear program. The indicator variable y_{ij} is one iff item j is assigned to i.

$$\begin{aligned} \text{maximize} \quad & \sum_{ij} b_{ij} y_{ij} & \\ \text{subject to} \quad & \sum_i y_{ij} \le 1 & \text{for all } j \\ & \sum_j b_{ij} y_{ij} \le B_i & \text{for all } i \\ & y_{ij} \in \{0,1\} & \end{aligned}$$

Why don't we require the bidders to adjust their bids to their left-over budget? Then we could not formulate the off-line problem. So the bids are something like the *value of item j to bidder i*. If this value changes over time, the model is inadequate.

As usual we obtain the linear programming relaxation by relaxing the integrality constaints on the y_{ij}'s to $y_{ij} \ge 0$. Upper bound constraints on the y_{ij}'s are already part of the system and therefore do not have to added.

The dual linear program has variables x_i for the bidders and z_j for the items.

$$\begin{aligned} \text{minimize} \quad & \sum_i B_i x_i + \sum_j z_j & \\ \text{subject to} \quad & b_{ij} x_i + z_j \ge b_{ij} & \text{for all } i \text{ and } j \\ & x_i \ge 0,\ z_j \ge 0 & \end{aligned}$$

In an optimal solution to the dual, $z_j = \max_i(1-x_i)b_{ij}$. If budgets are unlimited ($B_i = \infty$ for all i), any item can be assigned to the bidder i that bids highest for j. Then $x_i = 0$ for all i and $z_j = \max_i b_{ij}$ for all j in the optimal dual solution. If budgets are zero, no item can be assigned and $y_{ij} = 0$, $x_i = 1$, and $z_j = 0$ in optimal solutions. If the budgets are finite, the optima are somewhere in between.

Question 1 *Give intuitive interpretations for the x_i and z_j.*

LPs of the format above are called packing LPs and covering LPs. In the primal, we are trying to pack items into the budgets of the bidders subject to the constraint that items can be packet at most once and budgets must be observed. In the dual, the task is to cover the b_{ij}'s.

The authors came up with a very simple on-line algorithm that works well when

$$R_{\max} = \max_{ij} \frac{b_{ij}}{B_i}$$

is small. The algorithm follows the primal-dual paradigm. It constructs an almost feasible integral primal solution and a dual feasible solution such that

$$\begin{aligned}&\text{objective value of almost feasible integral primal solution}\\ &\qquad\qquad = (1-1/c)\cdot\text{objective value of dual feasible solution,}\end{aligned} \tag{1}$$

where $c = (1+R_{\max})^{1/R_{\max}}$. Observe that c goes to e (=2.71...) as $R_{\max}$ goes to zero (if $R_{\max} = 1$, $c = \sqrt{2}$, if $R_{\max} = 1/3$, $c = \sqrt[3]{4/3}$, ...). In a post-processing step, the almost feasible integral primal solution is converted into a feasible integral primal solution with

$$\begin{aligned}&\text{objective value of feasible integral primal solution}\\ &\qquad \geq (1-R_{\max})\cdot\text{objective value of almost feasible integral primal feasible solution.}\end{aligned} \tag{2}$$

Since the objective value of the dual feasible solution is at least the objective value of the optimal dual feasible solution which in turn is equal to the objective value of the optimal fractional primal feasible solution which in turn is at least the objective value of the optimal primal feasible solution, we obtain:

Theorem 1 ([BJN07]) *The on-line algorithms returns an integral feasible solution to the primal whose objective value is at least $(1-1/c)\cdot(1-R_{\max})$ times the objective value of the optimal integral feasible solution to the primal, i.e., the algorithm is $(1-1/c)\cdot(1-R_{\max})$-competitive.*

We next describe the algorithm.

1. Initialize all x_i to zero.
2. When item j arrives, let i maximize $b_{ij}(1-x_i)$. If $x_i \geq 1$, leave j unassigned, set $y_{ij} = 0$, $z_j = 0$, and leave all x_i unchanged.

3. Otherwise
   - set $y_{ij} = 1$, i.e., assign j to i.
   - set $z_j = b_{ij}(1 - x_i)$.
   - update x_i as follows:
   $$x_i = x_i(1 + \frac{b_{ij}}{B_i}) + \frac{b_{ij}}{(c-1)B_i}$$
   and leave the other x_ℓ unchanged.
4. After all items have arrived: let M_i be the items assigned to i. If $\sum_{j \in M_i} > B_i$ unassign the last item assigned to i.

The update rule for the x_i shows some similarity to a savings plan. The current capital x_i grows by an interest rate of b_{ij}/B_i and new capital $b_{ij}/((c-1)B_i)$ is added. We should expect some kind of exponential growth.

We come to the analyis. We prove:

- the dual solution is feasible.
- in any iteration:
$$\text{increase of cost of primal} = \frac{c-1}{c} \cdot \text{increase of cost of dual.} \tag{3}$$
- if $\sum_{j \in M_i} \geq B_i$, then $x_i \geq 1$, i.e., before the last assignment of an item to i, i was within its budget.

If j stays unassigned, $b_i j(1 - x_i) \leq 0$ for all i. Thus setting z_j to zero, satisfies the dual constraint for z_j. Otherwise, we set z_j to $\max_i b_{ij}(1 - x_i)$ and hence satisfy the dual constraint for z_j. The x_i's do not decrease in the course of the algorithms and hence constraints stay true. This takes care of the first bullet point.

If j is not assigned, the primal and the dual objective do not change. So assume that j is assigned to i. The objective of the primal increases by b_{ij}. The objective of the dual increases by

$$B_i \Delta x_i + z_j = B_i \left(\frac{b_{ij} x_i}{B_i} + \frac{b_{ij}}{(c-1)B_i} \right) + b_{ij}(1 - x_i) = b_{ij}(1 + \frac{1}{c-1}) = \frac{c}{c-1} b_{ij}.$$

Thus the objective of the primal increases by $(c-1)/c$ times the objective of the dual in any iteration. This takes care of the second item and establishes (1).

We turn to the third bullet point. Let M_i be the set of items assigned to i. Initially, M_i is empty. Let $V = \sum_{j \in M_i} b_{ij}$ be the total value of the items in M_i. We show

$$x_i \geq \frac{1}{c-1} \left(c^{V/B_i} - 1 \right).$$

Initially, $x_i = 0$ and $V = 0$ and the claim holds. When $V \geq B_i$, the right hand side is at least one and we have achieved our proof goal. Assume now that some item j is assigned to i. Then V grows to $V + b_{ij}$ and x_i changes to

$$\begin{aligned} x_i\left(1+\frac{b_{ij}}{B_i}\right)+\frac{b_{ij}}{(c-1)B_i} &\geq \frac{1}{c-1}\left(c^{V/B_i}-1\right)\left(1+\frac{b_{ij}}{B_i}\right)+\frac{b_{ij}}{(c-1)B_i} \\ &\geq \frac{1}{c-1}\left(c^{V/B_i}\left(1+\frac{b_{ij}}{B_i}\right)-1\right) \end{aligned}$$

It therefore suffices to show $(1 + b_{ij}/B_i) \geq c^{b_{ij}/B_i}$ or $\ln(1 + b_{ij}/B_i) \geq (b_{ij}/B_i) \cdot \ln c$. For $x > 0$ the function $x \mapsto \frac{\ln(1+x)}{x}$ is increasing (compute the derivative) and hence it suffices to show $\ln(1 + R_{\max}) \geq R_{\max} \cdot \ln c$. This holds by definition of c.

Before the last item is assigned to i, $x_i < 1$ and hence the value of the items assigned to i up to this point is at most B_i. Therefore undoing the last assignment if i's budget is exceeded will make the primal solution feasible. Let M_i be the items assigned to i before the postprocessing step. The value of the items assigned to i after the postprocessing step is at least

$$\sum_{j \in M_i} b_{ij}(1 - R_{\max}).$$

This holds true if no element is unassigned. If an element is unassigned $\sum_{j \in M_i} b_{ij} \geq B_i$ and the unassigned item has value at most $R_{\max} B_i$. This establishes (2) and completes the proof of the theorem.

You may wonder where the update rule for the x_i's comes from? If we are aiming for equality (3), the update rule is forced. Observe that the cost of the primal changes by b_{ij} and the cost of the dual changes by $B_i \Delta x_i + z_i$. If we aim for $b_{ij} = \frac{c-1}{c}(B_i \Delta x_i + z_i)$, we need to set

$$\Delta x_i = (\frac{c}{c-1} b_{ij} - z_j)\frac{1}{B_i} = \frac{1}{B_i}(\frac{c}{c-1} b_{ij} - b_{ij} + x_i b_{ij}) = \frac{b_{ij} x_i}{B_i} + \frac{b_{ij}}{(c-1)B_i}.$$

The paper states that the approximation ration $1 - 1/e$ is best possible. I have not checked this claim.

The fact that the update rule leads to exponential growth is suggested by its similarity to a savings plan.

Do we have to aim for (3)? We do not need it for every step, but we need it on average. At any point in time, let s_i be the fraction of i's budget that is spent at this point of time. Also assume that we set x_i to some increasing function f of s_i. We need $f(0) = 0$, because we initialize the x_i to zero, $f(1) \geq 1$, because we want that only one item needs to be removed in the postprocessing step. Let $b_1, b_2, \ldots, b_k$ be the values of the items assigned to i; I am suppressing the index i for simplicity. We want

$$\sum_j b_j \geq \alpha \sum_j (B_i(f(s_j) - f(s_{j-1})) + (1 - f(s_{j-1}))b_j)$$

for α as large as possible. Here $s_j = (b_1 + \ldots + b_j)/B_i$. Observe that the right hand side is the increase in the dual objective. Next observe that $s_j = s_{j-1} + b_j/B_i$ and $b_j/B_i \le R_{\max} \approx 0$. This suggests to replace $(f(s_j) - f(s_{j-1})$ by $f'(s_{j-1})b_j/B_i$, i.e., by the value of the derivative times the step size. The right hand side becomes

$$\begin{aligned}\alpha \sum_j (B_i f'(s_{j-1}) b_j/B_i + (1 - f(s_{j-1})b_j &= \alpha \sum_j (f'(s_{j-1}) + 1 - f(s_{j-1}))b_j \\ &\le \alpha \max_j (f'(s_{j-1}) + 1 - f(s_{j-1})) \sum_j b_j \\ &\le \sum_j b_j,\end{aligned}$$

for the choice $\alpha = 1/\max_{0 \le s \le 1}(f'(s) + 1 - f(s))$. In other words, we need a function f with $f(0) = 0$, $f(1) = 1$ and $\max_{0 \le s \le 1}(f'(s) + 1 - f(s))$ as small as possible. Recall that we want a large α.

We want to minimize $\max_{0 \le s \le 1}(f'(s) + 1 - f(s))$. What are the functions f for which the value $f'(s) + 1 - f(s)$ is constant, say β. The differential equation $f'(s) + 1 - f(s) = \beta$ has the general solution $f(s) = Ce^s + 1 - \beta$. The condition $f(0) = 0$ forces $C = \beta - 1$ and hence $f(s) = (\beta - 1)(e^s - 1)$. The condition $f(1) = 1$ forces $(\beta - 1)(e - 1) = 1$ and hence $\beta = e/(e - 1)$. Thus $\alpha = (e - 1)/e = 1 - 1/e$. This suggest that no better update rule exists.

References

[BJN07] Niv Buchbinder, Kamal Jain, and Joseph Naor. Online primal-dual algorithms for maximizing ad-auctions revenue. In Lars Arge, Michael Hoffmann, and Emo Welzl, editors, *ESA 2007*, volume 4698 of *Lecture Notes in Computer Science*, pages 253–264. Springer Berlin / Heidelberg, 2007.