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Computer Algebra
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To be handed in on April, 26th.
Discussion on April, 27th.

Exercise 2

2.1 Box functions

Let $f \in \mathcal{C}^\omega$, $f : \mathbb{R} \to \mathbb{R}$ be an analytic real function. Show that

$$\square f(I) := \left[f(m) - \sum_{k>0} \frac{\left|f^{(k)}(m)\right|}{k!} r^k,\ f(m) + \sum_{k>0} \frac{\left|f^{(k)}(m)\right|}{k!} r^k \right]$$

for intervals $I = [m - r,\ m + r] \subset \mathbb{R}$ (where $m \in \mathbb{R}$ and $r > 0$) defines a box function $\square f$ for f.

2.2 Real root isolation algorithm

Use the box function discussed in the previous exercise to devise an algorithm (based on EVAL) for the isolation of the real roots of a (squarefree) polynomial $f \in \mathbb{R}[x]$ in some interval $I_0 = [a, b] \subset \mathbb{R}$.

2.3 Implementation of a real root solver

Implement the algorithm you devised for the previous exercise in a programming language of your choice. We recommend a computer algebra system like Maple, Maxima or Sage, but feel free to use C, C++, Java or Python instead. For other languages, please ask in advance – your poor tutor might need some guidance.
Run your solver on the following instances and interpret the results:

1. $f_1(x) = \sum_{k=0}^{11} \frac{1}{k!} x^k, \quad I_1 = [-64, 64]$
2. $f_2(x) = \prod_{k=0}^{10} (x - k), \quad I_2 = [0, 32]$
3. $f_3(x) = x^{42} - (42x - 1)^2, \quad I_3 = [-2, 2]$
4. $f_4(x) = x^{42} + (42x + 1)^2, \quad I_4 = [-2, 2]$
5. $f_5(x) = ((x - \frac{2}{3})^6 - 1)(9x^4 - \frac{1}{9})((6x - 4)^3 + 1), \quad I_5 = [-8, 8]$

2.4 Subdivision tree width of EVAL

Show that for a squarefree polynomial $f \in \mathbb{R}[x]$ with degree $\deg f = n$, the subdivision tree width of your modified EVAL is bounded by $\mathcal{O}(n^2)$ nodes (corresponding to active intervals) at each subdivision level.

Hints:

1. Show that

$$\left|\frac{f^{(k)}(m)}{f(m)}\right| = \left|\sum_{i_1,\ldots,i_k}' \frac{1}{(m-z_{i_1})\cdots(m-z_{i_k})}\right| \leq \left(\sum_{i=1}^{n}\left|\frac{1}{m-z_i}\right|\right)^k \quad \text{for all } k \geq 1,$$

   where $z_1, \ldots, z_n$ are the (complex) roots of f and $\sum'$ denotes the sum over all tuples $(i_1, \ldots, i_k)$ with pairwise distinct entries i_j.

2. If $0 \leq a_k < \nu^k$ for all $k \geq 1$, then it holds that

$$\sum_{k \geq 1} a_k \frac{x^k}{k!} \leq e^{\nu x} - 1 \quad \text{for } x \geq 0.$$

3. Use the preceding inequalities to bound the distance of the midpoint m of an active interval with radius r to a (complex) root of f.

Have fun with the solution!