Computer Algebra
Michael Sagraloff

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To be handed in on May, 17th.
Discussion on May, 18th.

Exercise 5

5.1 A complex analogue to monotonicity for root isolation

Consider the following test $T_K^f(m,r)$ for a polynomial $f \in \mathbb{C}[z]$ and a real value $K \geq 1$ on a complex disc $D_r(m) := \{z \in \mathbb{C} : |z - m| \leq r\}$, $m \in \mathbb{C}$, $r \in \mathbb{R}_{>0}$:

$$T_K^f(m,r) \overset{def}{\iff} |f(m)| > K \sum_{k \geq 1} \left| \frac{f^{(k)}(m)}{k!} \right| r^k.$$

Prove: if $T_{\sqrt{2}}^{f'}(m,r)$ holds, then f has at most one root in $D_r(m)$. Proceed as follows:

1. Show that $T_K^f(m,r)$ implies $|f(z)| > (1 - \frac{1}{K})\,|f(m)|$. (In particular, as for the real case, which you considered in exercise 2.2, this shows that if $T_1^f(m,r)$ holds, then f has no root in $D_r(m)$.)

Now assume that $T_K^{f'}(m,r)$ holds for $K \geq \sqrt{2}$.

2. Prove that each root of f in $D_r(m)$ has multiplicity one.
3. Show by contradiction that there cannot be two distinct roots of f in $D_r(m)$.
   (a) Show that $\arg f'(m)$ and $\arg f'(z)$ differ by less than $\frac{\pi}{4}$ for an arbitrary $z \in D_r(m)$.
   (b) Assume that there are two distinct roots a, b of f in $D_r(m)$, $a \neq b$. By the Cauchy-Riemann differential equations and the Mean Value Theorem in two real variables, deduce that there are two points in $D_r(m)$ whose arguments differ by exactly $\frac{\pi}{2}$. Conclude that this contradicts (3a).

5.2 A complex root isolator

1. Suppose you are given a square-free polynomial $f \in \mathbb{C}[z]$ as well as a box $[a,b] = \{x + \mathrm{i}\,y : \operatorname{Re} a \leq x \leq \operatorname{Re} b \wedge \operatorname{Im} a \leq y \leq \operatorname{Im} b\} \subset \mathbb{C}$ which is known to comprise all complex roots of f.

   Use the T_K^f-test to devise and implement a subdivision-based root isolator for the roots of f.

   Hint: Recall exercises 2.2 and 2.4. In particular, note that if $T_1^f(m,r)$ does not hold, then f has a root in $D_{2nr}(m)$. Combine with the results of exercise 5.1 to decide whether a region of the complex plane needs further refinement.

2. **(Bonus)** What do you do if the initial box $[a,b]$ is not given?

   Hint: Exploit that f has exactly n distinct complex roots, where n is the degree of f.

5.3 Properties of (sub-)resultants

1. Show that resultants and subresultants are compatible with ring homomorphisms that preserve the degree. That is, for a homomorphism $\phi : R \to S$, its canonical extension $\Phi : R[x] \to S[x]$ and polynomials $f, g \in R[x]$, it holds that

$$\operatorname{Res}(\Phi f, \Phi g) = \Phi \operatorname{Res}(f, g) \qquad \text{and} \qquad \operatorname{Sres}(\Phi f, \Phi g) = \Phi \operatorname{Sres}(f, g),$$

   unless both leading coefficients of f and g vanish under ϕ, that is, $\phi(\operatorname{LCF} f) = \phi(\operatorname{LCF} g) = 0$.

2. Let f, g be bivariate polynomials in $\mathbb{C}[x, y] \cong (\mathbb{C}[x])[y]$, that is, we consider f and g as univariate polynomials over the ring $R = \mathbb{C}[x]$.

   Prove that $\operatorname{Sres}_k(f, g)$ can be represented as $\operatorname{Sres}_k(f, g) = uf + vg$ for some polynomials $u, v \in \mathbb{C}[x, y]$.

5.4 Resultants for projection of roots

Let $f, g \in \mathbb{C}[x, y] \cong (\mathbb{C}[x])[y]$ be bivariate polynomials.
Show that the roots of $\operatorname{Res}(f, g) \in \mathbb{C}[x]$ are exactly the union of the projections of the common solutions of $f = g = 0$ to the x-coordinate and the roots of the greatest common divisor $h = \gcd(\operatorname{LCF}_y(f), \operatorname{LCF}_y(g)) \in \mathbb{C}[x]$ of the leading coefficients of f and g in y:

$$\{x \in \mathbb{C} : \operatorname{Res}(f, g)(x) = 0\} = \mathcal{V}_{\mathbb{C}}^{(x)} \cup \{x \in \mathbb{C} : h(x) = 0\},$$

where $\mathcal{V}_{\mathbb{C}}^{(x)} = \{x \in \mathbb{C} : \exists y \in \mathbb{C} \text{ with } f(x, y) = g(x, y) = 0\}$.

Have fun with the solution!