Exercise 8

8.1 Horner’s rule
Let $R$ be a ring (commutative, with 1) and $u \in R$. Prove that Horner’s rule not only computes the remainder $f(u)$ of a polynomial $f \in R[x]$ on division by $x - u$, but also the coefficients of the quotient $(f - f(u))/(x - u)$.

8.2 Homogeneous bivariate polynomials
We call a bivariate polynomial $f(x, y)$ homogeneous if the degree of all terms in $f$ is the same. So $x^2 - y^2 - xy$ and $xy^3 - x^2y^2 + y^4$ are homogeneous, but $x^2 + y$ is not.

- Use http://exacus.mpi-inf.mpg.de/cgi-bin/xalci.cgi to plot the curves $x^2 - y^2 - xy = 0$, and $xy^3 - x^2y^2 + y^4 = 0$, and $x^2 - y^2 = 0$. Formulate a conjecture about the shape of vanishing sets of homogeneous polynomials.
- Show that any homogeneous polynomial factors into linear factors of the form $ax + by$ with $a, b \in \mathbb{C}$.

8.3 The shape of a curve near the origin
Let $f$ be a bivariate polynomial, and let $f^*$ be the homogeneous polynomial formed by the lowest order terms of $f$. For $f(x, y) = y^3 + x^2 - y^2 + 2xy$, $f^*$ consists of all terms of degree 2, that is, $f^* = x^2 - y^2 + 2xy$.

- Experiment with different $f$’s. Use http://exacus.mpi-inf.mpg.de/cgi-bin/xalci.cgi to plot the curves $f(x, y) = 0$ and $f^*(x, y) = 0$ near the origin. Formulate a conjecture.
- Prove the conjecture.

8.4 Chinese remaindering
Let $a_1, \ldots, a_r \in \mathbb{R}$ be pairwise distinct interpolation points with corresponding multiplicities $m_1, \ldots, m_r \in \mathbb{N}$ such that $\sum_{i=1}^{r} m_i = n + 1$.
Use the Chinese Remainder Theorem to show that, for each combination $b_{1,0}, \ldots, b_{1,m_1-1}, \ldots, b_{r,0}, \ldots, b_{r,m_r-1} \in \mathbb{R}$, there is a unique polynomial $f \in \mathbb{R}[x]$ of degree $n$ such that $f^{(j)}(a_i) = b_{i,j}$ for all $i = 1, \ldots, r$ and $j = 0, \ldots, m_i - 1$. 
8.5 Bounds on polynomial coefficients and root separation

(Bonus) Let \( f = \sum_{i=0}^{n} a_i x^i = a_n \prod_{j=1}^{n} (x - \zeta_j) \in \mathbb{Z}[x] \) be square-free, with (complex) roots \( \zeta_1, \ldots, \zeta_n \). We denote by \( \text{sep}(f) := \min \{|\zeta_i - \zeta_j| : i \neq j\} \) the minimum root separation of \( f \), that is the minimal distance between distinct complex roots.

Now assume that the bitlengths of the coefficients of \( f \) are bounded by \( \tau \in \mathbb{N} \), that is \( |a_i| \leq 2^\tau \) for all \( i, j \). Show that \( \text{sep}(f) \) is bounded from below by \( 2^{-\mathcal{O}(n(\tau+\log n))} \), that is there exists a \( c \in \mathbb{R}_{>0} \) such that \( \text{sep}(f) \geq 2^{-cn(\tau+\log n)} \).

**Hint 1:** Recall exercise 4.3 and, in particular, the idea of the proof of part (iii).

**Hint 2:** Use the fact that the Mahler measure \( \text{Mea}(g) := |b_n| \cdot \prod_{i=1}^{n} \max \{1, |\xi_i|\} \) of a polynomial \( g \in \mathbb{Z}[x] \) of degree \( n \) is bounded by \( \|g\| = \sqrt{b_0^2 + b_1^2 + \cdots + b_n^2} \) from above, where the \( \xi_i \)'s are the complex roots of \( g \) and the \( b_i \)'s its coefficients.

Have fun with the solution!