Computer Algebra
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To be handed in on June, 14th.
Discussion on June, 15th.

Exercise 9

9.1 Properties of Möbius transformations

In the lecture, we mentioned the group of Möbius transformations

$$\mathrm{Aut}(\overline{\mathbb{C}}) := \left\{ f : \overline{\mathbb{C}} \to \overline{\mathbb{C}},\ z \mapsto \frac{az+b}{cz+d} \text{ with } a,b,c,d \in \mathbb{C} \text{ and } \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0 \right\}$$

with the conventions $f(-\frac{d}{c}) = \infty$ and $f(\infty) = \frac{a}{c}$ for $c \neq 0$ and $f(\infty) = \infty$ for $c = 0$,

and the basic members

$$\begin{array}{rll} \text{translations:} & t_\lambda : \overline{\mathbb{C}} \to \overline{\mathbb{C}},\ z \mapsto z + \lambda & \text{for } \lambda \in \mathbb{C}, \\ \text{homothetic transformations:} & h_\lambda : \overline{\mathbb{C}} \to \overline{\mathbb{C}},\ z \mapsto \lambda z & \text{for } \lambda \in \mathbb{C} \setminus \{0\} \text{ and} \\ \text{reciprocal transformations:} & r : \overline{\mathbb{C}} \to \overline{\mathbb{C}},\ z \mapsto \dfrac{1}{z}. & \end{array}$$

In this exercise, we aim to show some basic properties of those Möbius transformations.

1. Show how each element $f \in \mathrm{Aut}(\overline{\mathbb{C}})$ can be decomposed into a concatenation of a finite number of the elementary operations.

For the following parts, we consider sets in $\mathbb{C}$ under the canonical representation of $\mathbb{C}$ as a real two-dimensional Euclidean plane $\mathbb{R}^2$. Prove:

2. t_λ and h_λ map lines to lines and circles to circles.

3. r maps lines through the origin to lines and all other lines to circles. Vice versa, circles are mapped to circles if they do not contain the origin, and otherwise are mapped to lines.

4. t_λ, h_λ and r preserve angles between lines and circles.

9.2 Implementation of a Descartes root solver

Implement a subdivision root solver based on Descartes' Rule of Signs. Run the Descartes solver on the instances from assignment sheet 2, exercise 3, and compare to your implementation of EVAL. How many subdivision steps do the two approaches require?

9.3 Subdivision in Descartes solvers

Let $f \in \mathbb{Z}[x]$ and $I = (a, b) \subset \mathbb{R}$ be some interval with midpoint $m = \frac{a+b}{2}$. Assume you are given the "localization" f_I of f in I as well as the intermediate polynomial $g_I(x) = f(a + x(b - a))$, occuring in the evaluation of the Möbius transformation of f to f_I.
We aim to show that, by storing the g_I's, the application of the bisection rule in a subdivision Descartes solver can be performed cheaper than computing f_{I_1} and f_{I_2} naïvely from scratch (that is, referring to f), where $I_1 := (a, m)$ and $I_2 := (m, b)$.

1. Show how to obtain g_{I_1} from g_I and g_{I_2} from g_{I_1}.
2. Show how to obtain f_{I_1} and f_{I_2} from g_{I_1} and g_{I_2}.

Only use the operations t_1, h_λ for $\lambda = 2^k$ for some $k \in \mathbb{Z}$ and r as components of the corresponding Möbius transformations.

Hint: What is $\varphi_I(1)$?

9.4 Descartes' Rule of Signs in Bernstein basis

A polynomial $F \in \mathbb{R}[x]$ of degree n in *Bernstein basis* with respect to some interval $[c, d] \subset \mathbb{R}$ is given by its coefficients with respect to the *Bernstein polynomials*

$$B_i^n[c, d] = \binom{n}{i} \frac{(x - c)^i (d - x)^{n-i}}{(d - c)^n},$$

that is, we write F as

$$F = \sum_{i=0}^{n} b_i \, B_i^n[c, d] = \sum_{i=0}^{n} b_i \binom{n}{i} \frac{(x - c)^i (d - x)^{n-i}}{(d - c)^n}.$$

1. Show that the Bernstein polynomials $(B_i^n[c, d])_{i=0,\ldots,n}$ form a basis of the $\mathbb{R}$-vector space of real polynomials of degree up to n.
2. Prove the *Descartes' Rule of Signs in Bernstein basis:* Let $I = (c, d)$ be an interval and $F = \sum_{i=0}^{n} b_i \, B_i^n[c, d]$ as above. Let m_I denote the number of real roots of F in I, counted with multiplicity. Then,

   $$v(F) := v(b_0, \ldots, b_n) \geq m_I \quad \text{and} \quad v(F) \equiv m_I \mod 2.$$

   Hint: Show $v(F) = v(F, I)$.

(Bonus) In the case you are familiar with Bézier curves: can you make up a geometric intuition why this holds?

Have fun with the solution!