12.1 Zariski topology

In this exercise, we establish the notion of the Zariski topology over $\mathbb{A}^n$ for algebraic varieties over some field $F$. It is induced by the definition of the closed sets as the algebraic sets in $\mathbb{A}^n$:

$$M \subset \mathbb{A}^n \text{ is Zariski-closed } \iff M = \mathcal{V}(I) := \mathcal{V}_{\mathbb{A}^n}(I) \text{ for some ideal } I \subset F[x_1, \ldots, x_n].$$

Let $C := \{\mathcal{V}(I) : I \subset F[x_1, \ldots, x_n] \text{ an ideal}\}$ denote the set of all Zariski-closed sets.

1. Let $O := \{\mathbb{A}^n \setminus C : C \in C\}$ be the Zariski-open sets. Show that $(\mathbb{A}^n, O)$ in fact defines a topological space:
   (a) Both the empty set and $\mathbb{A}^n$ are in $O$.
   (b) $O$ is closed under arbitrary union.
      (I.e., $O_j \in O$ for all $j \in J$ implies $\bigcup_{i \in J} O_i \in O$ for arbitrary $J$.)
   (c) $O$ is closed under finite intersection.
      (I.e., $O_j \in O$ for all $j \in \{1, \ldots, n\}$ implies $\bigcap_{1 \leq j \leq n} O_j \in O$ for $n \in \mathbb{N}$.)

2. We define the Zariski closure $\overline{M}$ of $M \subset \mathbb{A}^n$ as $\overline{M} := \mathcal{V}(I(M))$. Prove:

$$\mathcal{V}(I) \setminus \mathcal{V}(J) \subset \mathcal{V}(I : J) \text{ for ideals } I \text{ and } J \text{ in } F[x_1, \ldots, x_n].$$

3. Prove: If $F$ is algebraically closed and $I$ is radical, then equality holds: $\overline{\mathcal{V}(I) \setminus \mathcal{V}(J)} = \mathcal{V}(I : J)$.

12.2 Primary decomposition

1. Prove Lemma 7.1.5.

2. Show that if $Q_1$ and $Q_2$ are $P$-primary, then $Q_1 \cap Q_2$ is $P$-primary as well.

3. Show that $I : g^m = I : g^{m+1}$ implies that $I = (I : g^m) \cap (I, g^m)$.

4. Find (by hand) a primary decomposition for the radical of $I := (y^2 + yz, x^2 - xz, x^2 - z^2)$.
12.3 Gröbner basis computation

Let $I = \langle f_1, f_2 \rangle$, where

\[
\begin{align*}
    f_1 &= x^3y - 3x^2y^2 + x^2y - x^2 - 3xy^2 + 3y^3 + 6y \\
    f_2 &= x^2y + xy - x - 3y.
\end{align*}
\]

Compute a Gröbner basis for $I$ with respect to the lexicographical order, and determine all solutions of $f_1 = f_2 = 0$.

12.4 Point Sets

Let $X \subset \mathbb{Q}^n$ be a set of $d$ points such that the $n$-th coordinates of all these points are distinct. Show that the Lexicographic Gröbner Basis for $\mathcal{I}(X)$ is of the form

\[
\begin{align*}
    (x_1 - p_1(x_n), x_2 - p_2(x_n), \ldots, x_{n-1} - p_{n-1}(x_n), p_n(x_n)),
\end{align*}
\]

where $p_n$ is of degree $d$ and the other $p_i$ of degree at most $d - 1$. Hint: Use Lagrange interpolation!

Have fun with the solution!