Computer Algebra
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Summer 2011
To be handed in on July, 12th.
Discussion on July, 13th.

Exercise 12

12.1 Zariski topology

In this exercise, we establish the notion of the *Zariski topology* over $\mathbb{A}^n$ for algebraic varieties over some field F. It is induced by the definition of the closed sets as the algebraic sets in $\mathbb{A}^n$:

$$M \subset \mathbb{A}^n \text{ is Zariski-closed} \quad \overset{def}{\Leftrightarrow} \quad M = \mathcal{V}(I) := \mathcal{V}_{\mathbb{A}^n}(I) \text{ for some ideal } I \subset F[x_1, \ldots, x_n].$$

Let $\mathcal{C} := \{\mathcal{V}(I) : I \subset F[x_1, \ldots, x_n] \text{ an ideal}\}$ denote the set of all Zariski-closed sets.

1. Let $\mathcal{O} := \{\mathbb{A}^n \setminus C : C \in \mathcal{C}\}$ be the Zariski-open sets. Show that $(\mathbb{A}^n, \mathcal{O})$ in fact defines a topological space:
   (a) Both the empty set and $\mathbb{A}^n$ are in $\mathcal{O}$.
   (b) $\mathcal{O}$ is closed under arbitrary union.
   (I.e., $O_j \in \mathcal{O}$ for all $j \in J$ implies $\bigcup_{i \in J} O_i \in \mathcal{O}$ for arbitrary J.)
   (c) $\mathcal{O}$ is closed under finite intersection.
   (I.e., $O_j \in \mathcal{O}$ for all $j \in \{1, \ldots, n\}$ implies $\bigcap_{1 \le j \le n} O_j \in \mathcal{O}$ for $n \in \mathbb{N}$.)
2. We define the *Zariski closure* $\overline{M}$ of $M \subset \mathbb{A}^n$ as $\overline{M} := \mathcal{V}(\mathcal{I}(M))$. Prove:
$$\overline{\mathcal{V}(I) \setminus \mathcal{V}(J)} \subset \mathcal{V}(I : J) \quad \text{for ideals } I \text{ and } J \text{ in } F[x_1, \ldots, x_n].$$
3. Prove: If F is algebraically closed and I is radical, then equality holds: $\overline{\mathcal{V}(I) \setminus \mathcal{V}(J)} = \mathcal{V}(I : J)$.

12.2 Primary decomposition

1. Prove Lemma 7.1.5.
2. Show that if Q_1 and Q_2 are P-primary, then $Q_1 \cap Q_2$ is P-primary as well.
3. Show that $I : g^m = I : g^{m+1}$ implies that $I = (I : g^m) \cap (I, g^m)$.
4. Find (by hand) a primary decomposition for the radical of $I := (y^2 + yz, x^2 - xz, x^2 - z^2)$.

12.3 Gröbner basis computation

Let $I = \langle f_1, f_2 \rangle$, where

$$f_1 = x^3y - 3x^2y^2 + x^2y - x^2 - 3xy^2 + 3y^3 + 6y \quad \text{and}$$
$$f_2 = x^2y + xy - x - 3y.$$

Compute a Gröbner basis for I with respect to the lexicographical order, and determine all solutions of $f_1 = f_2 = 0$.

12.4 Point Sets

Let $X \subset \mathbb{Q}^n$ be a set of d points such that the n-th coordinates of all these points are distinct. Show that the Lexicographic Gröbner Basis for $\mathcal{I}(X)$ is of the form

$$(x_1 - p_1(x_n), x_2 - p_2(x_n), \ldots, x_{n-1} - p_{n-1}(x_n), p_n(x_n)),$$

where p_n is of degree d and the other p_i of degree at most $d-1$. Hint: Use Lagrange interpolation!

Have fun with the solution!