max planck institut
informatik

Computer Algebra
Michael Sagraloff

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To be handed in on July, 19th.
Discussion on July, 20th.

Exercise 13

13.1 Polynomial representation modulo the twisted cubic

Show that every polynomial $f \in \mathbb{C}[x, y, z]$ has a representation of the type

$$f = g_1(y - x^2) + g_2(z - x^3) + h,$$

where $g_1, g_2 \in \mathbb{C}[x, y, z]$ and $h \in \mathbb{C}[x]$.

13.2 Minimal Gröbner Bases

Prove Lemma 7.2.9 !

13.3 Solving systems of polynomial equations

Consider the system of equations

$$\begin{aligned} x^2 + y^2 + z^2 &= 4 \\ x^2 + 2y^2 &= 5 \\ xz &= 1 \end{aligned}$$

and the associated ideal $I \subset \mathbb{C}[x, y, z]$.

1. Compute Gröbner bases for $I_1 = I \cap \mathbb{C}[y, z]$ and $I_2 = \mathbb{C}[z]$.
2. How many solutions has this system of equations in $\mathbb{Q}^3$ or $\mathbb{R}^3$?

13.4 Complexity of Buchberger's algorithm

Give (coarse, but finite) upper bounds in the degree of the input polynomials for the number of iterations performed in Buchberger's algorithm using

1. the graded lexicographic order $>_{\text{grlex}}$ and
2. the pure lexicographic order $>_{\text{lex}}$.

13.5 Another version of multivariate division with remainder

(Bonus) The following considerations show that also more general division schemes lead to terminating division algorithms: Let $>$ be a global monomial order on $R = F[x_1, \ldots, x_n]$ for a field F and let $f_1, \ldots, f_r \in R \setminus \{0\}$.

1. Show that the following division process terminates:

   At each stage, we remove *some* term cx^α of the current dividend by reduction with some $\mathrm{In}(f_i)$ by which x^α is divisible. We stop as soon as this is no longer possible.

2. Assume that $\langle f_1, \ldots, f_r \rangle$ is a Gröbner basis of I. Show that the representation

   $$g = g_1 f_1 + \cdots + g_r f_r + h,$$

   for $g \in R$, resulting from the division with remainder as above, satisfies

   $$h = 0 \quad \text{if and only if} \quad g \in I = \langle f_1, \ldots, f_r \rangle.$$

Have fun with the solution!