Optimization I | **Summer 2011** | **Homework 1**

Introduction to Linear Optimization

This assignment is due on **April 28 in lecture**. You are allowed (even encouraged) to discuss these problems with your fellow classmates. All submitted work, however, must be *written individually* without consulting someone else's solutions or any other source like the web.

1. Consider the following LP:

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & -2x_1 + x_2 \leq 4 \\ & x_2 \leq 6 \\ & 3x_1 + x_2 \leq 18 \\ & x_2 \geq 0, x_1 \text{ unrestricted.} \end{aligned}$$

   (a) Plot the feasible region and find the optimal solution.

   (b) What is the optimal objective value if we remove the constraint $3x_1 + x_2 \leq 18$ from the LP?

   (c) Show how to reformulate the LP from (a) into the form minimize $c^T x$ subject to $Ax \geq b, x \geq 0$ (i.e., you need to give appropriate vectors c and b and matrix A).

2. (Problem 1.4 in B & T) Consider the problem

$$\begin{aligned} \text{minimize} \quad & 2x_1 + 3|x_2 - 10| \\ \text{subject to} \quad & |x_1 + 2| + |x_2| \leq 5, \end{aligned}$$

   and reformulate it as a linear program.

3. Let $P = \{x \in \mathbb{R}^n : Ax \geq b\}$, and suppose P is bounded and non-empty. Let $x^1, \ldots, x^k$ be the vertices of P. In class, we will prove that any vector z in P can be written as a convex combination of $x^1, \ldots, x^k$, i.e. for any $z \in P$ there exist non-negative scalars $\lambda_1, \ldots, \lambda_k$ with $\sum_{i=1}^k \lambda_k = 1$ such that $z = \sum_{i=1}^k \lambda_i x^i$.
   Use this fact to prove that for any $c \in \mathbb{R}^n$, the LP

$$\min c^T x \text{ subject to } Ax \geq b$$

   has an optimal solution that is a vertex of P.

   Note that this proves that if P is bounded and non-empty, we can find an optimal solution by checking all vertices of P.

4. (Extra Credit) Consider the LP:

$$\begin{aligned} \text{minimize} \quad & c^T x + d^T y \\ \text{subject to} \quad & Ax + By = b \\ & x \geq 0, y \text{ unrestricted.} \end{aligned}$$

   Suppose we want to convert this problem to one in standard form; that is, with all variables being nonnegative. In class, we saw that one could do this by replacing the unconstrained variables with the difference of nonnegative variables (e.g. $x_j = x_j^+ - x_j^-$ for $x_j^+ \geq 0, x_j^- \geq 0$), but this doubles the number of such variables. Devise another technique to obtain an equivalent standard form problem where the number of variables is only increased by one.