| Optimization I | Summer 2011<br>Homework 3 |
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# The Simplex Method

This assignment is due on May 17 in lecture. You are allowed (even encouraged) to discuss these problems with your fellow classmates. All submitted work, however, must be *written individually* without consulting someone elses solutions or any other source like the web.

1. (B&T 3.18) Consider the simplex method applied to a standard form problem and assume that the rows of the matrix $A$ are linearly independent. For each of the following statements, give either a proof or a counterexample.
   (a) Prove that a variable that has just left the basis cannot reenter in the very next iteration.
   (b) Give an example that shows that a variable that has just entered the basis can leave in the very next iteration.

2. (B&T 3.19) While solving a problem in standard form, we arrive at the following tableau, with $x_3, x_4$ and $x_5$ being the basic variables:

| | | | | | |
|---|---|---|---|---|---|
| -10 | $\delta$ | -2 | 0 | 0 | 0 |
| 4 | -1 | $\eta$ | 1 | 0 | 0 |
| 1 | $\alpha$ | -4 | 0 | 1 | 0 |
| $\beta$ | $\gamma$ | 3 | 0 | 0 | 1 |

   The entries $\alpha, \beta, \gamma, \delta, \eta$ in the tableau are unknown parameters. For each one of the following statements, find some parameter values that will make the statement true.
   (a) The current solution is optimal and there are multiple optimal solutions.
   (b) The optimal cost is $-\infty$.
   (c) The current solution is feasible but not optimal.

3. Suppose we are solving $\min c^T x$ s.t. $Ax = b, x \geq 0$, with $A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ -1 & 0 & 2 & 1 \\ -1 & 1 & 0 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 0 \\ \frac{3}{2} \end{bmatrix}$, and $c^T = [2, 1, 3, 1]$.

   To find an initial basic feasible solution, we introduced artificial variables $y^T = [y_1, y_2, y_3]$ and a new objective function $\min \sum_{i=1}^{3} y_i = [1, 1, 1] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ and we solve the auxiliary (Phase I) problem: $\min \sum_{i=1}^{3} y_i$ s.t. $Ax + Iy = b, x \geq 0, y \geq 0$.
   (a) Use the simplex method to solve the auxiliary problem. Your first tableau should have $y_1, y_2, y_3$ as the basic variables, and you should use Bland's rule to choose the entering and leaving variables.

   If you did part (a) correctly, then your final tableau should have $x_2, x_3$ and $y_3$ as the basic variables.

   If we drop the $y$'s, then we have found a feasible solution to $Ax = b, x \geq 0$, but it is not a basic feasible solution, since we have only two basic variables, but our matrix $A$ has three rows, so we should have three basic variables. So we would like to perform an additional pivot so that some real variable $x_j$ enters the basis and $y_3$ leaves the basis.

(b) Explain why, in this case, performing a pivot that lets $x_j$ enter the basis and $y_3$ leave the basis does not increase the objective value, even if $\bar{c}_j \geq 0$.

In fact, we can choose $y_3$ as leaving variable even if the *pivot element* is negative. You can think about this as follows: normally, when we let $x_j$ enter the basis, this means we are moving in the direction $d$ which has $d_j = 1, d_{j'} = 0$ for $j' \neq j, B(1), \ldots, B(m)$ and $d_B = -A_B^{-1}A_j$. Recall that $-d_B$ is equal to the $j$-th column of the tableau, and that we denoted this by $u$. We then compute by how much we can move in direction $d$: we need $x + \theta d \geq 0$ and (since $d_k \geq 0$ for all non-basic variables $x_k$) we find that $\theta^* = \min_{i:u_i>0} x_{B(i)}/u_i$. Now, suppose we moved in the direction $-d$ instead of direction $d$. We cannot really move in direction $-d$, since this would cause $x_j$ to become negative. However, if we pretend we do move in direction $-d$, and compute the value $\theta^*$ as $\min_{i:u_i<0} |x_{B(i)}/u_i|$, then we find $\theta^* = 0$ because $y_3$ is a basic variable that is zero. So in this special case, we can move in the direction $-d$ to a new basis. **Note that we only ever do this at the end of Phase I to force the artificial variables out of the basis!**

So, we can just look for a column $j$ corresponding to a real variable $x_j$, for which $(3, j)$ is not 0, and we let $x_j$ enter the basis and the artificial variable leaves the basis. We use elementary row operations to change column $j$ to $\begin{bmatrix} 0 \\ e_3 \end{bmatrix}$.

(b) Give the new tableau if you let $x_4$ leave the basis and $y_3$ enter the basis.

(c) Now, all artificial variables are non-basic, and we have found a basic feasible solution $x$ to the original problem. We remove the corresponding columns from the tableau, and we need to put back the real objective: we thus need to compute the reduced costs $\bar{c}_j = c_j - c_B^T A_B^{-1} A_j$ for the current basis matrix $A_B$.

It turns out we can read off $A_B^{-1}$ from the final tableau of Phase I. Explain which part of the tableau contains $A_B^{-1}$, and give the reduced cost we need to start Phase II for our example.

(d) Now suppose we were doing the Phase I simplex method to find an initial basic feasible solution to some general LP in standard form, i.e. the feasible region is given by $Ax = b, x \geq 0$. Suppose that Phase I terminates and that there is still some artificial variable $y_i$ that is basic, and suppose that the $(i, j)$-th entry is 0 for all real variables $x_j$. Prove that this means that the rows of $A$ are linearly dependent.

One can show that in this case, the $i$-th constraint is redundant, and we can thus drop the $i$-th row of the tableau. The end of Phase I can thus be summarized as follows: (i) pivot artificial variable $y_i$ out of the basis, if there exists a real variable $x_j$ such that element $(i, j)$ in the tableau is non-zero, (ii) for all remaining artificial variables $y_i$, drop the corresponding row of the tableau.